Review of Wave Equation

Moving waves carry energy from one point to another. Waves have velocity (light waves in vacuum propagate with \(3 \times 10^8 \text{ m/sec}\); sound waves in air \(330 \text{ m/sec}\).)

Some waves (most EM and sound waves) exhibit linearity as the total of two linear waves is the sum of the two waves as they would exist separately.

(i) Plane waves: characterized by a disturbance that at a given (free space) point of time has uniform properties across an infinite plane perpendicular to the direction of propagation.

(ii) Cylindrical waves: uniform across cylindrical surface.

(iii) Spherical waves: uniform across spherical surface.
A medium is lossless if it doesn't attenuate the amplitude of the wave traveling within it or on its surface.

\[ y(x,t) = A \cos \left( \frac{2\pi}{T} - \frac{2\pi x}{\lambda} + \phi_0 \right) \]  

amplitude \hspace{1cm} \text{time period} \hspace{1cm} \text{spatial wavelength} \hspace{1cm} \text{reference phase}

Phase equivalent expression:

\[ y(x,t) = A \cos \phi(x,t) \]

\[ \phi(x,t) = \left( \frac{2\pi}{T} - \frac{2\pi x}{\lambda} + \phi_0 \right) \sim \text{instantaneous phase} \]  

\( 2\pi \text{ radians} = 360^\circ \)

The wave pattern repeats itself at a spatial period \( \lambda \) along \( x \) and at a temporal period \( T \) along \( t \).

The phase velocity: \( v_p = \frac{\lambda}{T} = \frac{2\pi}{2\pi} = 1 \) (propagation velocity is the velocity of the wave pattern as it moves).

If one of the terms is positive and the other is negative, then \(-\) positive \( x \)-direction; if both are positive or negative, then \(-\) negative \( x \)-direction.

Frequency: \( f = \frac{1}{T} \) (Hz), \( \Rightarrow \) \( v_p = \frac{\lambda}{T} \) (m/s)

\[ y(x,t) = A \cos \left( 2\pi f t - \frac{2\pi}{\lambda} x \right) \]

Angular velocity: \( \omega = 2\pi f \) (rad/s)

Phase constant (wavenumber): \( \beta = \frac{2\pi}{\lambda} \) (rad/m) \( \Rightarrow \) \( y(x,t) = A \cos (\omega t - \beta x) \)

(Propagation to \(-\) \( x \)-direction) \( \Rightarrow \) \( y(x,t) = A \cos (\omega t - \beta x + \phi_0) \)

Lossy Medium: \( y(x,t) = Ae^{-\alpha x} \cos (\omega t - \beta x + \phi_0) \)

\( \alpha \): attenuation factor \( (\text{m}^{-1}) \)
Complex Numbers

\[ z = x + jy , \quad x = \text{Re}(z) , \quad y = \text{Im}(z) \]

Polar form: \[ z = |z| e^{j\theta} = |z| e^{j\theta} \]

Euler's Identity: \( e^{j\theta} = \cos(\theta) + j\sin(\theta) \)

\[
(x = |z| \cos(\theta) , \quad y = |z| \sin(\theta) \]
\[
|z| = \sqrt{x^2 + y^2} , \quad \theta = \tan^{-1}(y/x) \]

Complex conjugate: \( z^* = (x-jy) = x-jy = |z| e^{-j\theta} \)

\[
\begin{align*}
z_1 + z_2 &= (x_1 + x_2) + j(y_1 + y_2) \\
z_1 - z_2 &= (x_1 - x_2) + j(y_1 - y_2) \\
z_1 \cdot z_2 &= |z_1||z_2| e^{j(\theta_1 + \theta_2)} , \quad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)} \\
z^n = |z|^n e^{j\theta n} = |z|^n (\cos(n\theta) + j\sin(n\theta)) \\
z^{1/n} &= |z|^{1/n} e^{j\theta/n} \\
-1 &= e^{j\pi} = 1 \angle \pi \\
j &= e^{j\pi/2} = 1 \angle \pi/2 \\
-j &= e^{-j\pi/2} = 1 \angle -\pi/2 \\
\end{align*}
\]

Phasors are useful for single-frequency waves. \( \sin(x) = \sin(\pi x) \)

(Adopt a cosine reference) \( \cos(\pi x) = \cos(\pi x - \pi) \)

\[
\begin{align*}
V_s(t) &= V_0 \sin(\omega t + \phi_0) = V_0 \cos(\frac{\pi}{2} - \omega t - \phi_0) = V_0 \cos(\omega t + \phi_0) \\
v(t) &= \text{Re} \left[ z e^{j\omega t} \right] = v_s(t) = \text{Re} \left[ V_0 e^{j(\omega t + \phi_0 - \pi/2)} \right] = \text{Re} \left[ V_0 e^{j\phi_0} e^{j\omega t} \right] = \text{Re} \left[ V_0 e^{j\phi_0} \right] \\
V_s &= V_0 e^{j(\phi_0 - \pi/2)} \\
\end{align*}
\]
Transmission Lines

\[ V_{AA'} = V_g(t) = V_0 \cos(\omega t) \quad (v) \]

\[ V_{BB'} = V_{AA'} \left( k - \frac{t}{c} \right) = V_0 \cos \left[ \omega \left( t - \frac{t}{c} \right) \right] \]

Phase Difference: \( \frac{\omega l}{c} = \frac{2\pi f l}{c} = \frac{2\pi f l}{c} \text{ rad} \)

1. \( l/c < \lambda \) \( \Rightarrow \) Turbine effects can be ignored \( \Leftrightarrow \) low frequencies
2. \( l/c \geq \lambda \) \( \Rightarrow \) Consider turbine effects

Reflections, power loss, dispersion

\[ \lambda = \frac{v}{f} \]

Propagation Modes

1. TEM modes: Electric and Magnetic fields are entirely transverse to the direction of propagation.

2. Quasi-TEM modes: non-transverse field components are \( \ll \) transverse field components.

TEM Lines: consist of two separate conducting surfaces

Higher-order modes: At least one significant field component in the direction of propagation

\[ \useMaxwell's\ eq. \]
Parallel-wire configuration regardless of the specific shape of the line under consideration (Applicable to TEM lines)

(A) Lossless (TEM line)

L': combined inductance of both conductors per unit length (H/m)
C': capacitance of two conductors per unit length (F/m)

\[
\begin{align*}
L' \frac{\partial^2 V(t)}{\partial t^2} + j \omega L' I(t) &= U(t) \\
C' \frac{\partial^2 I(t)}{\partial t^2} - j \omega C' V(t) &= 0
\end{align*}
\]

Telegrapher's equations

\[
\begin{align*}
\text{Phasors: } & U(t) = \text{Re} \{ V(t) e^{j\omega t} \}, \quad I(t) = \text{Re} \{ I(t) e^{j\omega t} \} \\
- \frac{1}{j \omega L'} \frac{dV(t)}{dt} &= j \omega L' I(t), \quad - \frac{1}{j \omega C'} \frac{dI(t)}{dt} = j \omega C' \frac{dV(t)}{dt}
\end{align*}
\]

For TEM lines: \( \beta = \omega \sqrt{L'C'} \), \( U_p = \frac{1}{\beta} \)

\[ \frac{d^2 V(t)}{dt^2} + \beta^2 V(t) = 0 \]

\( V(t) = V_0 e^{-\beta t} + V_0 e^{\beta t} \)

For TEM lines: \( \beta = \omega \sqrt{\mu_0 \varepsilon_0} \) (max.), \( U_p = \frac{1}{\beta} \) (m/s)

\( \varepsilon = \varepsilon_0 \varepsilon \), \( \mu = \mu_0 \)

\( \lambda = \frac{U_p}{f} = \frac{C_0}{\sqrt{\varepsilon \varepsilon_0}} \)
R': Combined resistance of both conductors per unit length.

C': Conductance of insulating medium per unit length.

\( \frac{\partial v(t)}{\partial t} = R' i(t) + \frac{\partial i(t)}{\partial t} \) \quad \rightarrow \quad \frac{\partial i(t)}{\partial t} = C' \frac{\partial v(t)}{\partial t}

\( \Delta v(t) \) \quad \rightarrow \quad \Delta v(t) = (i' + j\omega C') \Delta v(t)

\( \frac{d^2 \Delta v(t)}{d \tau^2} - (R' + j\omega L') (C' + j\omega C') \Delta v(t) = 0 \)

Complex propagation constant: \( \gamma = \sqrt{(R' + j\omega L') (C' + j\omega C')} \)

\( \gamma = \alpha + j\beta \)

Attenuation constant \( \neq \) lossless

\( a = Re(\gamma), \quad \beta = Im(\gamma), \quad \alpha = \frac{\omega}{\beta}, \quad \lambda = \frac{\omega}{\beta} \)

For TEM lines:

\( \frac{R'}{L'} = \frac{G'}{C'} \quad \text{TEM condition} \)
Example 1

Coax Line:

\[ L' = \frac{\mu}{2\pi} \ln \left( \frac{b}{a} \right) \]
\[ C' = \frac{2\pi \varepsilon}{\ln \left( \frac{b}{a} \right)} \]

Ridge coaxial line with inner conductor diameter of 0.6 cm and outer conductor diameter of 1.2 cm. Conductors made of copper.

\[ \varepsilon = \varepsilon_0 \varepsilon_r, \quad \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m}, \quad \varepsilon_r = 1 \]

Conductivity: \[ \sigma = 5.8 \times 10^7 \text{ S/m}, \quad \mu_0 = 4\pi \times 10^{-3} \text{ H/m} \]

\[ L' = \frac{\mu_0}{2\pi} \ln \left( \frac{b}{a} \right) = 0.14 \text{ (nH/m)} \]

\[ C' = \frac{2\pi \varepsilon_0 \varepsilon_1}{\ln \left( \frac{b}{a} \right)} = 80.3 \text{ (pF/m)} \]

Surface resistance of metal:

\[ R' = \frac{R_s \left( \frac{1}{a} + \frac{1}{b} \right)}{2\pi} \]

\[ R_s = \sqrt{\frac{2\pi}{\sigma}}, \quad \sigma = 5.8 \times 10^7 \text{ S/m} \]

\[ G' = \frac{2\pi \sigma}{\ln \left( \frac{b}{a} \right)} \]

\[ \sigma = 0 \quad \text{(conductors (insul. \ \varepsilon_0 = \varepsilon_r))} \]
Example 2

Two wire line

\[ E(z) = \frac{V_0}{z_0} e^{-j \beta z} + \frac{V_0}{z_0} e^{j \beta z} \]

\[ \beta = \frac{2 \pi}{\lambda} \]

Separated by distance \( 2a \), wire and earth is along line.

Conductors with \( \varepsilon = \infty \).

Given \( \mu = \infty \), \( \frac{R_1}{R_2} = \frac{R_0}{1} \),

\[ R_1 = \frac{R_0}{1} \]

\[ \mu = \frac{\sqrt{\mu}}{2} \]

\[ \mu = \frac{120}{\mu} \]

\[ \mu = \frac{120}{1} \]

\[ \mu = 120 \]

To solve for \( R_1 \).
2-1 General Considerations

In most electrical engineering curricula, the study of electromagnetics is preceded by one or more courses on electrical circuits. In this book, we use this background to build a bridge between circuit theory and electromagnetic theory. The bridge is provided by transmission lines, the topic of this chapter. By modeling the transmission line in the form of an equivalent circuit, we can use Kirchhoff’s voltage and current laws to develop wave equations whose solutions provide an understanding of wave propagation, standing waves, and power transfer. Familiarity with these concepts facilitates the presentation of material in later chapters.

Although the family of transmission lines may encompass all structures and media that serve to transfer energy or information between two points, including nerve fibers in the human body, acoustic waves in fluids, and mechanical pressure waves in solids, we shall focus our treatment in this chapter on transmission lines used for guiding electromagnetic signals. Such transmission lines include telephone wires, coaxial cables carrying audio and video information to TV sets or digital data to computer monitors, and optical fibers carrying light waves for the transmission of data at very high rates. Fundamentally, a transmission line is a two-port network, with each port consisting of two terminals, as illustrated in Fig. 2-1. One of the ports is the sending end and the other is the receiving end. The source connected to its sending end may be any circuit with an output voltage, such as a radar transmitter, an amplifier, or a computer terminal operating in the transmission mode. From circuit theory, any such source can be represented by a Thévenin-equivalent generator circuit consisting of a generator voltage \( V_g \) in series with a generator resistance \( R_g \), as shown in Fig. 2-1. The generator voltage may consist of digital pulses, a modulated time-varying sinusoidal signal, or any other signal waveform. In the case of a-c signals, the generator circuit is represented by a voltage phasor \( V_g \) and an impedance \( Z_g \).

![Diagram of transmission line with components labeled: \( R_s \), \( A \), \( V_s \), \( A' \), \( R_L \), \( B \), \( B' \)].

Figure 2-1: A transmission line is a two-port network connecting a generator circuit at the sending end to a load at the receiving end.
Figure 2-4: A few examples of transverse electromagnetic (TEM) and higher-order transmission lines.

Figure 2-5: In a coaxial line, the electric field lines are in the radial direction between the inner and outer conductors, and the magnetic field forms circles around the inner conductor.
Table 2-1: Transmission-line parameters $R'$, $L'$, $G'$, and $C'$ for three types of lines.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coaxial</th>
<th>Two Wire</th>
<th>Parallel Plate</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R'$</td>
<td>$\frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right)$</td>
<td>$\frac{R_s}{\pi a}$</td>
<td>$\frac{2R_s}{\omega}$</td>
<td>$\Omega/m$</td>
</tr>
<tr>
<td>$L'$</td>
<td>$\frac{\mu_s}{2\pi} \ln(b/a)$</td>
<td>$\frac{\mu_s}{\pi a} \left[ \ln \left( \frac{d}{2a} \right) + \sqrt{\left( \frac{d}{2a} \right)^2 - 1} \right] - \frac{\sigma w}{\omega}$</td>
<td>$\frac{\sigma w}{\omega}$</td>
<td>$H/m$</td>
</tr>
<tr>
<td>$G'$</td>
<td>$\frac{2\pi \sigma}{\ln(b/a)}$</td>
<td>$\frac{\pi \sigma}{\ln(b/a)} \left[ \ln \left( \frac{d}{2a} \right) + \sqrt{\left( \frac{d}{2a} \right)^2 - 1} \right]$</td>
<td>$\frac{\sigma w}{d}$</td>
<td>$S/m$</td>
</tr>
<tr>
<td>$C'$</td>
<td>$\frac{2\pi \varepsilon}{\ln(b/a)}$</td>
<td>$\frac{\pi \varepsilon}{\ln(b/a)} \left[ \ln \left( \frac{d}{2a} \right) + \sqrt{\left( \frac{d}{2a} \right)^2 - 1} \right]$</td>
<td>$\frac{\varepsilon w}{d}$</td>
<td>$F/m$</td>
</tr>
</tbody>
</table>

Notes: (1) Refer to Fig. 2-4 for definitions of dimensions. (2) $\mu$, $\varepsilon$, and $\sigma$ pertain to the insulating material between the conductors. (3) $\mu_s = \sqrt{\mu_0 \mu}$. (4) $\mu_0$ and $\sigma$ pertain to the conductors. (5) If $(d/2a)^2 \gg 1$, then $\ln \left( \frac{d}{2a} \right) + \sqrt{\left( \frac{d}{2a} \right)^2 - 1} \approx \ln(d/a)$.

2-5 THE LOSSLESS TRANSMISSION LINE

Table 2-2: Characteristic parameters of transmission lines.

<table>
<thead>
<tr>
<th></th>
<th>Propagation Constant $\gamma = \alpha + j\beta$</th>
<th>Phase Velocity $u_p = \omega/\beta$</th>
<th>Characteristic Impedance $Z_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>General case</td>
<td>$\gamma = \sqrt{\left( R' + j\omega L' \right) \left( G' + j\omega C' \right)}$</td>
<td>$u_p = \omega/\beta$</td>
<td>$Z_0 = \frac{(R' + j\omega L')}{(G' + j\omega C')}$</td>
</tr>
<tr>
<td>Lossless $(R' = G' = 0)$</td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_0/c}$</td>
<td>$u_p = c/\sqrt{\varepsilon_0}$</td>
<td>$Z_0 = \sqrt{L'/C'}$</td>
</tr>
<tr>
<td>Lossless coaxial</td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_0/c}$</td>
<td>$u_p = c/\sqrt{\varepsilon_0}$</td>
<td>$Z_0 = (60/\sqrt{\varepsilon_0}) \ln(b/a)$</td>
</tr>
<tr>
<td>Lossless two wire</td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_0/c}$</td>
<td>$u_p = c/\sqrt{\varepsilon_0}$</td>
<td>$Z_0 = (120/\sqrt{\varepsilon_0}) \ln(d/2a)$, if $d \gg a$</td>
</tr>
<tr>
<td>Lossless parallel plate</td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_0/c}$</td>
<td>$u_p = c/\sqrt{\varepsilon_0}$</td>
<td>$Z_0 = (120\varepsilon_0/\omega)(d/w)$</td>
</tr>
</tbody>
</table>

Notes: (1) $\mu = \mu_0$, $\varepsilon = \varepsilon_0\varepsilon$, $c = 1/\sqrt{\mu_0\varepsilon_0}$, and $\sqrt{\mu_0/\varepsilon_0} \approx (120\varepsilon_0/\omega)$ $\Omega$, where $\varepsilon_0$ is the relative permittivity of insulating material. (2) For coaxial line, $a$ and $b$ are radii of inner and outer conductors. (3) For two-wire line, $a$ = wire radius and $d$ = separation between wire centers. (4) For parallel-plate line, $w$ = width of plate and $d$ = separation between the plates.
LECTURE 3

General (lossy) transmission

\[ \tilde{V}(z) = V_0^+ e^{-z^2} + V_0^- e^{z^2} \]
\[ \tilde{I}(z) = I_0^+ e^{-z^2} + I_0^- e^{z^2} = \frac{Z}{R+jwL} \left[ V_0^+ e^{-z^2} - V_0^- e^{z^2} \right] \]

by replacing \( \tilde{V}(z) \) in:

\[ - \frac{d\tilde{I}(z)}{dz} = (\gamma' + jwC') \tilde{V}(z) \]

\[ \gamma' = \sqrt{(R+jwL') (\gamma + jwC')} \]

Define the characteristic impedance

\[ Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R+jwL'}{\gamma} = \sqrt{\frac{R+jwL'}{\gamma + jwC'}} \quad \text{(lossless)} \]

\[ Z_0 = \frac{\sqrt{L'}}{\sqrt{C}} \]

→ the ratio of the voltage to the current amplitude for each of the traveling waves individually (with an additional \((-1) \times -Z\)).

IT IS NOT EQUAL TO THE RATIO OF TOTAL VOLTAGE TO THE TOTAL CURRENT, UNLESS ONE OF THE TWO WAVES IS ABSENT!!

\[ \tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-z^2} - \frac{V_0^-}{Z_0} e^{z^2} \]
Voltage Reflection Coefficient

\[ Z_0 \quad \square \quad \text{Load} \quad Z_L \]

\[ Z_L = \frac{V_L}{I_L}, \quad V_L = V(z=0) = V_0^+ + V_0^- \]
\[ I_L = I(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \]

\[ V_0^- = \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) V_0^+ = - \frac{I_0^-}{I_0^+} \]

\[ \Rightarrow \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = - \frac{I_0^-}{I_0^+} \]

1. Always: \( |\Gamma| \leq 1 \)
2. Load is matched to the line if \( Z_L = Z_0 = \Gamma = 0 \) (no reflection)
3. Load: open circuit (\( Z_L = \infty \)) \( \Rightarrow \Gamma = 1 \)
4. Load: short circuit (\( Z_L = 0 \)) \( \Rightarrow \Gamma = 0 \)
5. Load: purely reactive (\( Z_L = jX_L \)) \( \Rightarrow |\Gamma| = 1, \Gamma \in \mathbb{C} \)

\[ V(z) = \frac{V_0^+ (e^{-j\frac{2\pi}{2}} + \Gamma e^{j\frac{2\pi}{2}})}{Z_0} \quad \text{(Standing wave)} \]

Define voltage standing wave ratio

\[ VSWR = S = \frac{|\bar{V}|_{\text{max}}}{|\bar{V}|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{as measure of the mismatch of load/line} \]

- Matched load: \( \Gamma = 0 \Rightarrow S = 1 \)
- \( |\Gamma| = 1 \Rightarrow S = \infty \quad \text{Always} \quad S \geq 1 \)

Power: \( P_{av} = \left| V_0^+ \right|^2 / Z_0, \quad P_{av}^r = \left| V_0^- \right|^2 / Z_0 = \left( \frac{V_0^-}{Z_0} \right)^2 = \left( \frac{I_0^-}{I_0^+} \right)^2 Z_0 = |\Gamma|^2 P_{av}^r \)
Example

\[ Z_L = (100 + j50) \, \Omega \]

\[ r = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50} = \frac{20.7 + \frac{48}{158.1}^{\circ}}{158.1 + \frac{138}{158.1}^{\circ}} \approx 0.45 \angle 26.6^{\circ} \]

\[ S = \frac{1 + r}{1 - r} = \frac{1 + 0.45}{1 - 0.45} = 2.6 \]

Input impedance of the lossless line

\[ Z_{in}(\xi) = \frac{\tilde{Y}(\xi)}{\tilde{Z}(\xi)} = Z_o \left[ \frac{1 + r e^{-j2\beta \xi}}{1 - r e^{-j2\beta \xi}} \right] \]

In terms of total voltage (incident-reflected) to the total current at any point \( \xi \) of the transmission line with characteristic impedance \( Z_0 \) and load \( Z_L \).

\[ Z_{in}(\xi) = Z_o \left[ \frac{1 + r e^{-j2\beta \xi}}{1 - r e^{-j2\beta \xi}} \right] = Z_o \left( \frac{Z_L \cos \beta \xi + j Z_0 \sin \beta \xi}{Z_o \cos \beta \xi + j Z_0 \sin \beta \xi} \right) = Z_o \left( \frac{Z_L + j Z_0 \tan \beta \xi}{Z_o + j Z_L \tan \beta \xi} \right) \]

\[ \Delta = Z_o \left( \frac{Z_L + j Z_0 \tan \beta \xi}{Z_o + j Z_L \tan \beta \xi} \right) \]
Example of (3)

\[ f = 1.05 \text{ GHz} \] \[ \Rightarrow f_c = 10.2 \]

\[ Z_i = (100 + j50) \Omega, \quad \theta_i = 50^\circ \quad \text{(default value)} \]

connected through a 6 cm-long Coaxial TRL Teflon for 2.64
Phase velocity on the line: \[ 0.7 c_0 = 0.7 \times 3 \times 10^8 \text{ m/s} \]

Is it matched to the generator?

Need to find: \( Z_{in} (Z = -0.67 m) \)

\[ \lambda = \frac{V_p}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2 \text{ m} \]

\[ l = \frac{\lambda}{2} \]

\[ l = k \lambda \]

\[ l = \frac{167}{2} \lambda = 3.35 \lambda \]

\[ \beta l = \frac{2\pi}{\lambda} \]

\[ \beta l = \frac{2\pi}{0.2} \]

\[ Z_{in} = Z_0 \left[ \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right] = \frac{100 + j50 + j50 \tan (6.2 \pi)}{50 + (100 + j50) \tan (6.2 \pi)} \]

\[ \text{Reflected Power} = |P_{in}|^2 P_{in} \]

\[ = (21.9 + j17.4) \Omega \neq Z_i \]

- Short-Circuited Load: \( Z = -1 \) \( \Rightarrow Z_{in} = jZ_0 \tan \beta l \) \( \rightarrow \) Pure Inductive

- \( \tan \beta l = 0 \) \( \Rightarrow \) Equivalent Inductance: \( j\omega L_{eq} = jZ_0 \tan \beta l \)

- \( \tan \beta l = 0 \) \( \Rightarrow \) Equivalent Capacitance: \( -j\omega C_{eq} = jZ_0 \tan \beta l \)

- Open-Circuited Load: \( Z_{in} = -jZ_0 \cot \beta l \)
A network analyzer is an RF instrument that measures the impedance of any load connected to its input terminal.

\[ Z_0 = \sqrt{\frac{Z_{in}}{Z_{out}}} \quad \text{(Determination of } Z_0 \text{ through } Z_{in}, Z_{out}) \]

**Quarter-wave Transformer**

**Line Specification**

\[ \tan \theta L = \tan \theta \frac{n}{4} \Rightarrow \theta L = \frac{n}{4} \pm m \frac{2\pi}{l} \]

\[ Z_{in}(Z = n \frac{\lambda}{4}) = Z_L \]

\[ l = \frac{\lambda}{4} + n \frac{\lambda}{4} \quad Z_{in} = Z_0 \frac{Z_L}{Z_0} \]

**Example**

\[ Z_L = 50 \Omega \rightarrow Z_{in} = 100 \Omega \]

Any transformer with \( Z_0 \):

\[ \frac{Z_0}{Z_L} = \frac{50}{100} = 0.5 \]

lossless line \( \rightarrow Z_0 \) real, math only real loads w/ \( Z_L \)

practical works for only 1 freq. xfr.
\[
\begin{align*}
\frac{dy}{dx} + y = \frac{1}{x} \\
\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} - y \\
\text{Integrating both sides, we get:} \\
\int \frac{dy}{dx} = \int \left( \frac{1}{x} - y \right) dx \\
\Rightarrow \quad y = \ln|x| + C \\
\text{Where } C \text{ is the constant of integration.}
\end{align*}
\]
\[ i = 0.47 \text{ Y} \]

\[ 4 + 1 = 41 \]
\[ 6 + 1 = 41 \]

At the two intersection points, the normalised displacement are

\[ \text{Figure 47: Solution to Example 47} \]

\[ \text{(d) Sides drawn at the same side} \]

\[ \text{Explain the solution, l, which results in the frequency variation of the material.} \]

\[ \text{Figure 46: Solution to Example 46} \]

\[ \text{(a) Sides drawn at the same side} \]

\[ \text{(b) Sides drawn at the same side} \]
The Quarter-Wave Transformer

\[ \frac{V_1}{V_2} = \sqrt{2} \]

where \( V_1 \) and \( V_2 \) are the voltages at the input and output, respectively.

When the transformer is lossless, the input and output voltages are equal in magnitude but opposite in phase.

The voltage on the primary is given by:

\[ V_1 = \sqrt{2} V_2 \]

The voltage on the secondary is given by:

\[ V_2 = \sqrt{2} V_1 \]

The ratio of the voltages is therefore:

\[ \frac{V_1}{V_2} = \sqrt{2} \]

The transformer is considered lossless if the ratio of the voltages is constant.

\[ \frac{V_1}{V_2} = \frac{V_{1p}}{V_{2p}} = \sqrt{2} \]

The transformer is considered lossy if the ratio of the voltages is not constant.

\[ \frac{V_{1p}}{V_{2p}} = \frac{V_1}{V_2} = \sqrt{2} \]

The transformer is considered perfect if the ratio of the voltages is constant and equal to 1.

\[ \frac{V_{1p}}{V_{2p}} = \frac{1}{\sqrt{2}} \]

The transformer is considered imperfect if the ratio of the voltages is not constant and equal to 1.

\[ \frac{V_{1p}}{V_{2p}} = \frac{1}{\sqrt{2}} \]

The transformer is considered ideal if the ratio of the voltages is constant and equal to 1.

\[ \frac{V_{1p}}{V_{2p}} = 1 \]

The transformer is considered non-ideal if the ratio of the voltages is not constant and equal to 1.

\[ \frac{V_{1p}}{V_{2p}} \neq 1 \]

The transformer is considered infinite if the ratio of the voltages is constant and equal to infinity.

\[ \frac{V_{1p}}{V_{2p}} = \infty \]

The transformer is considered finite if the ratio of the voltages is not constant and equal to infinity.

\[ \frac{V_{1p}}{V_{2p}} \neq \infty \]

The transformer is considered ideal if the ratio of the voltages is constant and equal to 1.

\[ \frac{V_{1p}}{V_{2p}} = 1 \]

The transformer is considered non-ideal if the ratio of the voltages is not constant and equal to 1.

\[ \frac{V_{1p}}{V_{2p}} \neq 1 \]
The numerator function is \( \frac{1}{2} \). The denominator function is \( y_1(2 \cos \theta + \cos \omega t) \). The frequency function is \( \omega = \frac{2\pi}{T} \). The displacement function is \( y_1 = A \sin \omega t \). The phase angle function is \( \theta = \frac{2\pi}{T} \).

\[ \begin{align*}
\frac{1}{\sqrt{2}} \cos \left( \frac{\pi}{2} + \frac{\pi}{2} \right) & = 0 \\
\cos \left( \frac{\pi}{2} + \frac{\pi}{2} \right) & = 0 \\
\cos \left( \frac{\pi}{2} + \frac{\pi}{2} \right) & = 1
\end{align*} \]

The result function is \( \frac{1}{\sqrt{2}} \).
The theory of small reflection

\[
\begin{align*}
\text{The total reflection in a composite mirror} & = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{1 - \frac{1}{n} + \frac{1}{n^2}} \\
\text{where} \quad n & = \text{index of refraction of the medium}
\end{align*}
\]

The diagram shows the reflection of light from a mirror.
transmission coefficients are

\[ \Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad \Gamma_2 = -\Gamma_1, \quad \Gamma_3 = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T_{12} = 1 + \Gamma_1 = \frac{2Z_1}{Z_1 + Z_2}, \quad T_{21} = 1 + \Gamma_3 = \frac{2Z_2}{Z_1 + Z_2}. \]

We can compute the total reflection, \( \Gamma \), seen by the feed line by the impedance method or by the multiple reflection method, as discussed in Section 3.5. For our present purpose the latter technique is preferred, so we can express the total reflection as an infinite sum of partial reflections and transmissions as follows:

\[
\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_1 e^{-2j\theta} + T_{12}T_{21}\Gamma_1^2 e^{-4j\theta} + \cdots \\
= \Gamma_1 + T_{12}T_{21}\Gamma_1 e^{-2j\theta} + \sum_{n=0}^{\infty} T_{12}^n T_{21}^n \Gamma_1^2 e^{-2j\theta n},
\]

Using the geometric series,

\[ \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \text{for } |z| < 1, \]

(6.39) can be expressed in closed form as

\[ \Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_1 e^{-2j\theta}}{1 - T_{12}T_{21}\Gamma_1 e^{-2j\theta}}. \]

From (6.35), (6.37), and (6.38), we use \( \Gamma_1 = -\Gamma_1, T_{12} = 1 + \Gamma_1, \) and \( T_{21} = 1 - \Gamma_1 \) in (6.40) to give

\[ \Gamma = \Gamma_1 + \frac{\Gamma_1 e^{-2j\theta}}{1 - \Gamma_1 e^{-2j\theta}}. \]

Now if the discontinuities between the impedances \( Z_1, Z_2 \) and \( Z_3, Z_4 \) are small, then \( |\Gamma, \Gamma'| \ll 1, \) so we can approximate (6.41) as

\[ \Gamma \approx \Gamma_1 + \frac{\Gamma_1 e^{-2j\theta}}{1 - \Gamma_1 e^{-2j\theta}}. \]

This result states the intuitive idea that the total reflection is dominated by the reflection from the initial discontinuity between \( Z_1 \) and \( Z_2 \) (\( \Gamma_1 \)), and the first reflection from the discontinuity between \( Z_2 \) and \( Z_4 \) (\( \Gamma' = e^{-2j\theta} \)). The \( e^{-2j\theta} \) term accounts for the phase delay when the incident wave travels up and down the line. The following example demonstrates the accuracy of this approximation.

**EXAMPLE 6.6**

Consider the quarter-wave transformer of Figure 6.13, with \( Z_1 = 100 \, \Omega, \) \( Z_2 = 150 \, \Omega, \) and \( Z_L = 225 \, \Omega. \) Evaluate the worst-case percent error in computing \( |\Gamma| \) from the approximate expression of (6.42).

**Solution**

The partial reflection coefficients from (6.34) and (6.36) are

\[ \Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{150 - 100}{150 + 100} = 0.2, \]
\[ \Gamma_2 = -\Gamma_1 = -0.2, \]
\[ T_{12} = \Gamma_1^2 = 0.04. \]

Since the approximate expression for \( \Gamma \) in (6.42) is identical to the numerator for the exact expression in (6.41), the greatest error will occur when the denominator of (6.41) differs from unity to the greatest extent. This occurs for \( \theta = 0 \) or \( 90^\circ \).

Thus, for \( \theta = 90^\circ \) both results are zero. Then (6.41) gives the exact result as \( \Gamma = 0.384 \), while (6.42) gives the approximate result as \( \Gamma = 0.4. \) The error is about 4%.
Chapter 6: Impedance Matching and Tuning

Multisection Transformer

Now consider the multisection transformer shown in Figure 6.14. This transformer consists of \( N \) equal-length (commensurate) sections of transmission lines. We will derive an approximate expression for the total reflection coefficient \( \Gamma \).

Partial reflection coefficients can be defined at each junction, as follows:

\[
\Gamma_0 = \frac{Z_0 - Z_1}{Z_0 + Z_1} \quad \text{6.43a}
\]

\[
\Gamma_n = \frac{Z_{n-1} - Z_n}{Z_{n-1} + Z_n}, \quad n = 2, 3, \ldots, N \quad \text{6.43b}
\]

\[
\Gamma_N = \frac{Z_N - Z_{N-1}}{Z_N + Z_{N-1}} \quad \text{6.43c}
\]

We also assume that all \( Z_n \) increase or decrease monotonically across the transformer, and that \( Z_1 \) is real. This implies that all \( \Gamma_n \) will be real, and of the same sign (\( \Gamma_n > 0 \) if \( Z_n > Z_{n-1} \); \( \Gamma_n < 0 \) if \( Z_n < Z_{n-1} \)). Then using the results of the previous section, the overall reflection coefficient can be approximated as

\[
\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-j\theta} + \Gamma_2 e^{-j2\theta} + \cdots + \Gamma_N e^{-jN\theta} \quad \text{6.44}
\]

Further assume that the transformer can be made symmetrical, so that \( \Gamma_0 = \Gamma_N, \Gamma_1 = \Gamma_{N-1}, \Gamma_2 = \Gamma_{N-2}, \ldots \). Then (6.44) can be written as

\[
\Gamma(\theta) = e^{-jN\theta} \Gamma_0 (e^{jN\theta} + e^{-jN\theta} + \Gamma_1 e^{j(N-1)\theta} + e^{-j(N-1)\theta} + \cdots) \quad \text{6.45}
\]

If \( N \) is odd, the last term is \( \Gamma_{N/2} e^{j(N-1)\theta} + e^{-j(N-1)\theta} \), while if \( N \) is even the last term is \( \Gamma_{N/2} \). Equation (6.45) is then seen to be of the form of a finite Fourier cosine series in \( \theta \), which can be written as

\[
\Gamma(\theta) = 2 e^{-jN\theta} \Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2\theta) + \cdots + \Gamma_n \cos(N-2n\theta) + \cdots + \Gamma_N \cos(N-2N\theta) \quad \text{6.46a}
\]

\[
\Gamma(\theta) = 2 e^{-jN\theta} \Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2\theta) + \cdots + \Gamma_n \cos(N-2n\theta) + \cdots + \Gamma_N \cos(N-2N\theta) \quad \text{for } N \text{ odd.} \quad \text{6.46b}
\]

Figure 6.14 Partial reflection coefficients for a multisection matching transformer.

6.6 Binomial Multisection Matching Transformers

The importance of these results lies in the fact that we can synthesize any desired reflection coefficient response as a function of frequency \( \theta \), by properly choosing the \( \Gamma_n \) and using enough sections \( N \). This should be clear from the realization that a Fourier series can represent an arbitrary smooth function, if enough terms are used. In the next two sections we will show how to use this theory to design multisection transformers for two of the most commonly used passband responses: the binomial (maximally flat) response, and the Chebyshev (equal ripple) response.
actual response as given (approximately) by (6.44):

\[ \Gamma(\theta) = A \sum_{n=0}^{N} C_n A e^{-j\theta n} = \Gamma_0 + \Gamma_1 e^{-j\theta} + \Gamma_2 e^{-2j\theta} + \cdots + \Gamma_N e^{-jN\theta} \]

This shows that the \( \Gamma_n \) must be chosen as

\[ \Gamma_n = AC_n N \]

where \( A \) is given by (6.49), and \( C_n N \) is a binomial coefficient.

At this point, the characteristic impedances \( Z_n \) can be found via (6.43), but a simpler solution can be obtained using the following approximation [1]. Since we assumed that the \( \Gamma_n \) are small, we can write

\[ \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} = \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n} \]

since \( \ln x \approx 2(x - 1/e + 1) \). Then, using (6.52) and (6.49) gives

\[ \ln \frac{Z_{n+1}}{Z_n} = 2\Gamma_n = 2AC_n N \left( 2z_n - Z_n \right) Z_{n+1} Z_n = 2^{-N} C_n N \ln \frac{Z_1}{Z_0} \]

which can be used to find \( Z_{n+1} \) starting with \( n = 0 \). These results are approximate, but generally give usable results for \( 0.5Z_n < Z_L < 2Z_0 \).

Exact results can be found by using the transmission line equations for each section and numerically solving for the characteristic impedances [2]. The results of such calculations are listed in Table 6.1, which give the exact line impedances for \( N = 2, 3, 4, 5 \), and 6 section binomial matching transformers, for various ratios of load impedance, \( Z_L \), to feed line impedance, \( Z_0 \). The table gives results only for \( Z_L/Z_0 > 1 \); if \( Z_L/Z_0 < 1 \), the results for \( Z_0/Z_L \) should be used, but with \( Z_L \) starting at the load end. This is because the solution is symmetric about \( Z_L/Z_0 = 1 \); the same transformer that matches \( Z_L \) to \( Z_0 \) can be reversed and used to match \( Z_0 \) to \( Z_L \). More extensive tables can be found in reference [2].

The bandwidth of the binomial transformer can be evaluated as follows. As in Section 6.4, let \( \Gamma_n \) be the maximum value of reflection coefficient that can be tolerated over the passband. Then from (6.48),

\[ \Gamma_n = 2^n |A| \cos^n \theta_n \]

where \( \theta_n = \pi/2 \) is the lower edge of the passband, as shown in Figure 6.11. Thus,

\[ \theta_n = \cos^{-1} \left( \frac{\Gamma_n}{A} \right)^{1/N} \]

and using (6.53) gives the fractional bandwidth as

\[ \frac{\Delta f}{f_0} = \frac{2\pi f_0 - \Delta f}{f_0} = 2 - \frac{4\theta}{\pi} \]

\[ = 2 - \frac{4}{\pi} \cos^{-1} \left( \frac{\Gamma_n}{A} \right)^{1/N} \]

6.55
Design a three-section binomial transformer to match a 50 Ω load to a 100 Ω line, and calculate the bandwidth for \( \Gamma_m = 0.05 \). Plot the reflection coefficient magnitude versus normalized frequency for the exact designs using 1, 2, 3, 4, and 5 sections.

**Solution**
For \( N = 3 \), \( Z_L = 50 \) Ω, \( Z_0 = 100 \) Ω we have, from (6.49),

\[
A = 2^{-N} \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) = 2^{-3} \left( \frac{50 - 100}{50 + 100} \right) = 0.0417.
\]

From (6.55) the bandwidth is

\[
\Delta f = \frac{2}{A} \frac{4}{\pi} \cos^{-1} \left( \frac{\Gamma_m}{2} \right) = \frac{2}{0.0417} \frac{4}{\pi} \cos^{-1} \left( \frac{0.5}{2} \right) = 0.71 \text{ or } 71\%.
\]

The necessary binomial coefficients are

\[
C_0^3 = \frac{3!}{3!0!} = 1, \\
C_1^3 = \frac{3!}{2!1!} = 3, \\
C_2^3 = \frac{3!}{1!2!} = 3.
\]

Then using (6.53) gives the required characteristic impedances as

\[
n = 0: \quad \ln Z_1 = \ln Z_0 + 2^{-N} C_0^3 \ln \frac{Z_L}{Z_0} = \ln 100 + 2^{-3} (1) \ln \frac{50}{100} = 4.518, \\
Z_1 = 91.7 \Omega.
\]

\[
n = 1: \quad \ln Z_2 = \ln Z_1 + 2^{-N} C_1^3 \ln \frac{Z_L}{Z_0} = \ln 91.7 + 2^{-3} (3) \ln \frac{50}{100} = 4.26, \\
Z_2 = 70.7 \Omega.
\]

\[
n = 2: \quad \ln Z_3 = \ln Z_2 + 2^{-N} C_2^3 \ln \frac{Z_L}{Z_0} = \ln 70.7 + 2^{-3} (3) \ln \frac{50}{100} = 4.00, \\
Z_3 = 54.5 \Omega.
\]

To use the data in Table 6.1, we reverse the source and load impedances and consider the problem of matching a 100 Ω load to a 50 Ω line. Then \( Z_2/Z_0 = 2.0 \), and we obtain the exact characteristic impedances as \( Z_1 = 91.7 \) Ω, \( Z_2 = 70.7 \) Ω, and \( Z_3 = 54.5 \) Ω, which agree with the approximate results to three significant digits. Figure 6.15 shows the reflection coefficient magnitude versus frequency for exact designs using \( N = 1, 2, 3, 4, \) and 5 sections. Observe that greater bandwidth is obtained for transformers using more sections.

---

**6.7 CHEBYSHEV MULTISECTION MATCHING TRANSFORMERS**

In contrast with the binomial matching transformer, the Chebyshev transformer optimizes bandwidth at the expense of passband ripple. If such a passband characteristic can be tolerated, the bandwidth of the Chebyshev transformer will be substantially better than that of the binomial transformer, for a given number of sections. The Chebyshev transformer is designed by equating \( \Gamma(0) \) to a Chebyshev polynomial, which has the optimum characteristics needed for this type of transformer. Thus we will first discuss the properties of the Chebyshev polynomials, and then derive a design procedure for Chebyshev matching transformers using the small reflection theory of Section 6.5.

**Chebyshev Polynomials**

The nth order Chebyshev polynomial is a polynomial of degree \( n \), and is denoted by \( T_n(x) \). The first four Chebyshev polynomials are

\[
T_1(x) = x, \\
T_2(x) = 2x^2 - 1.
\]
\[ T_2(x) = 4x^2 - 3x, \quad T_3(x) = 8x^3 - 8x^2 + 1. \]

Higher-order polynomials can be found using the following recurrence formula:

\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \]

The first four Chebyshev polynomials are plotted in Figure 6.16, from which the following very useful properties of Chebyshev polynomials can be noted:

- For \(-1 \leq x \leq 1\), \(|T_n(x)| \leq 1\). In this range, the Chebyshev polynomials oscillate between \pm 1. This is the equal ripple property, and this region will be mapped to the passband of the matching transformer.
- For \(|x| > 1\), \(|T_n(x)| > 1\). This region will map to the frequency range outside the passband.
- For \(|x| > 1\), the \(|T_n(x)|\) increases faster with \(x\) as \(n\) increases.

Now let \(x = \cos \theta\) for \(|x| < 1\). Then it can be shown that the Chebyshev polynomials can be expressed as

\[ T_n(\cos \theta) = \cos n\theta, \]

or more generally as

\[ T_n(\cos \theta) = \cos(n \cos^{-1} x), \quad \text{for } |x| < 1, \]

\[ T_n(\cos \theta) = \cos(n \cos^{-1} x), \quad \text{for } |x| \geq 1. \]

We desire equal ripple in the passband of the transformer, so it is necessary to map \(\theta_n\) to \(x = 1\) and \(-\theta_n\) to \(x = -1\), where \(\theta_n\) and \(-\theta_n\) are the lower and upper edges of the passband, as shown in Figure 6.11. This can be accomplished by replacing \(\cos \theta\) in (6.58a) with \(\cos \theta\) of \(\theta_n\):

\[ T_n(\cos \theta) = T_n(\sec \theta_n \cos \theta) = \cos n \left[ \frac{\cos \theta}{\cos \theta_n} \right] \]

Then \(|\sec \theta_n \cos \theta| \leq 1\) for \(\theta_n < \theta < \pi - \theta_n\), so \(|T_n(\sec \theta_n \cos \theta)| \leq 1\) over this same range.

Since \(\cos \theta\) can be expanded into a sum of terms of the form \(\cos n\theta - 2m\theta\), the Chebyshev polynomials of (6.56) can be rewritten in the following useful form:

\[ T_n(\sec \theta_n \cos \theta) = \sec \theta_n \cos \theta, \quad T_n(\sec \theta_n \cos \theta) = \sec^2 \theta_n(1 + \cos 2\theta) - 1, \]

\[ T_n(\sec \theta_n \cos \theta) = \sec^3 \theta_n(\cos 3\theta + 3 \cos \theta) - 3 \sec \theta_n \cos \theta, \]

\[ T_n(\sec \theta_n \cos \theta) = \sec^4 \theta_n(\cos 4\theta + 4 \cos 2\theta + 3), \]

\[-4 \sec^2 \theta_n(\cos 2\theta + 1) + 1.\]

The above results can be used to design matching transformers with up to four sections, and will also be used in later chapters for the design of directional couplers and filters.

**Design of Chebyshev Transformers**

We can now synthesize a Chebyshev equal-ripple passband by making \(\Gamma(\theta)\) proportional to \(T_n(\sec \theta_n \cos \theta)\), where \(N\) is the number of sections in the transformer. Thus, using (6.46),

\[ \Gamma(\theta) = 2e^{-j/2N} \sum_{n=0}^{N-1} T_n(\sec \theta_n \cos \theta) \]

\[ = 2e^{-j/2N} T_N(\sec \theta_N \cos \theta), \]

where the last term in the series of (6.61) is \((1/2)T_{N/2}\) for \(N\) even and \(T_{N-1/2}\) for \(N\) odd. As in the binomial transformer case, we can find the constant \(A\) by letting \(\theta = 0\), corresponding to zero frequency. Thus,

\[ \Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_N(\sec \theta_n), \quad \cos(\theta) \approx 1, \]

so we have

\[ \frac{A}{Z_L + Z_0} = \frac{Z_L - Z_0}{T_N(\sec \theta_n)} \]

or

\[ A = \frac{Z_L - Z_0}{Z_L + Z_0} T_N(\sec \theta_n). \]

6.7 Chebyshev Multisection Matching Transformers
Now if the maximum allowable reflection coefficient magnitude in the passband is $\Gamma_m$, then from (6.61) $\Gamma_m = A$, since the maximum value of $\Gamma_m(\sec \theta_m \cos \theta)$ in the passband is unity. Then from (6.62) $\theta_m$ is determined as

$$
\theta_m = \cos^{-1} \left( \frac{1}{N} \frac{Z_l - Z_0}{Z_l + Z_0} \right)
$$

or, using (5.58b),

$$
\sec \theta_m = \cosh \left( \frac{1}{N} \cosh^{-1} \left( \frac{1}{N} \frac{Z_l - Z_0}{Z_l + Z_0} \right) \right)
$$

Once $\theta_m$ is known, the fractional bandwidth can be calculated from (6.33) as

$$
\frac{\Delta f}{f_0} = \frac{2 - \sec \theta_m}{\pi}
$$

From (6.61), the $\Gamma_m$ can be determined using the results of (6.60) to expand $T_m(\sec \theta_m \cos \theta)$ and equating similar terms of the form $\cos(N - 2\theta)$. The characteristic impedances $Z_0$ can then be found from (6.43). This procedure will be illustrated in Example 6.8.

The above results are approximate because of the reliance on small reflection theory, but are general enough to design transformers with an arbitrary ripple level, $\Gamma_m$. Table 6.2 gives exact results [2] for a few specific values of $\Gamma_m$, for $N = 2, 3$, and 4 sections; more extensive tables can be found in reference [2].

EXAMPLE 6.8

Design a third-section Chebyshev transformer to match a 100 $\Omega$ load to a 50 $\Omega$ line, with $\Gamma_m = 0.05$, using the above theory. Plot the reflection coefficient magnitude versus normalized frequency for exact designs using 1, 2, 3, and 4 sections.

Solution

From (6.61) with $N = 3$,

$$
\Gamma(\theta) = 2e^{-2\theta M}(\Gamma_0 \cos \theta + \Gamma_1 \cos \theta) = Ae^{-2\theta M}(\sec \theta_m \cos \theta).
$$

Then, $A = \Gamma_m = 0.05$, and from (6.63),

$$
\sec \theta_m = \cosh \left( \frac{1}{N} \cosh^{-1} \left( \frac{1}{N} \frac{Z_l - Z_0}{Z_l + Z_0} \right) \right)
$$

$$
= \cosh \left( \frac{1}{3} \cosh^{-1} \left( \frac{1}{100 - 50} \right) \right)
$$

$$
= 1.395
$$

So, $\theta_m = 44.2^\circ$.

Using (6.60c) for $T_2$ gives

$$
2(\Gamma_0 \cos \theta + \Gamma_1 \cos \theta) = A \sec ^2 \theta_m (\cos \theta + 3 \cos \theta) - 3A \sec \theta_m \cos \theta.
$$

Table 6.2

<table>
<thead>
<tr>
<th>$N = 2$</th>
<th>$\Gamma_m = 0.05$</th>
<th>$\Gamma_m = 0.20$</th>
<th>$\Gamma_m = 0.05$</th>
<th>$\Gamma_m = 0.20$</th>
</tr>
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<td>$Z_3/Z_0$</td>
<td>$Z_0/Z_0$</td>
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<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
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<td>1.3219</td>
<td>1.2247</td>
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</tr>
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<td>1.6402</td>
<td>1.3161</td>
<td>1.5197</td>
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<td>1.6456</td>
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</tr>
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<td>2.1561</td>
<td>2.5558</td>
</tr>
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<td>3.1721</td>
<td>3.6461</td>
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<td>4.6393</td>
<td>4.1862</td>
<td>4.2993</td>
</tr>
</tbody>
</table>

Equating similar terms in $\cos \theta M$ gives the following results:

$$
\cos \theta M = 2\Gamma_0 = A \sec ^2 \theta_m,
$$

$$
\Gamma_0 = 0.0678,
$$

$$
\cos \theta M = 3A(\sec ^2 \theta_m - \sec \theta_m),
$$

$$
\Gamma_0 = 0.099.
$$

From symmetry we also have that

$$
\Gamma_3 = \Gamma_0 = 0.0678,
$$

and

$$
\Gamma_3 = \Gamma_0 = 0.099.
$$

Then the characteristic impedances are

$$
Z_0 = 1 + \frac{\Gamma_m}{\Gamma_0} = 50 + 1.0678 = 57.27 \Omega,
$$

$$
1.0678.
$$
FIGURE 6.17  Reflection coefficient magnitude versus frequency for the multisection matching transformers of Example 6.8.

\[
Z_1 = Z_0 \left(1 + \frac{f_1}{f_0}\right) = 57.27 \left(1 + 0.099\right) = 69.86 \Omega, \\
Z_2 = Z_0 \left(1 - \frac{f_1}{f_0}\right) = 100 \left(1 - 0.0678\right) = 87.30 \Omega.
\]

\[\frac{\Delta r}{\Delta f} = 2 - \frac{44.2}{\pi} = 2 - 4.42 = 1.02\]

or 102%. This is significantly greater than the bandwidth of the binomial transformer of Example 6.7 (71%), which was for the same type of mismatch. The trade-off, of course, is a nonzero ripple in the passband of the Chebyshev transformer.

Figure 6.17 shows reflection coefficient magnitudes versus frequency for the exact designs from Table 6.2 for \(N = 1, 2, 3,\) and 4 sections.

6.8 TAPERED LINES

In the preceding sections we discussed how an arbitrary real load impedance could be matched to a line over a desired bandwidth by using multisection matching transformers.

As the number, \(N,\) of discrete sections increases, the step changes in characteristic impedance between the sections become smaller. Thus, in the limit of an infinite number of sections, we approach a continuously tapered line. In practice, of course, a matching transformer must be of finite length, often no more than a few sections long. But instead of discrete sections, the line can be continuously tapered, as suggested in Figure 6.18a.

Then by changing the type of taper, we can obtain different passband characteristics.

In this section we will derive an approximate theory, based on the theory of small reflections, to predict the reflection coefficient response as a function of the impedance taper, \(Z(z).\) We will then apply these results to a few common types of tapers.

Consider the continuously tapered line of Figure 6.18a as being made up of a number of incremental sections of length \(\Delta z\), with an impedance change \(\Delta Z(z)\) from one section to the next, as shown in Figure 6.18b. Then the incremental reflection coefficient from the step at \(z\) is given by

\[
\frac{\Delta r}{\Delta z} = \frac{(z + \Delta Z) - z}{(z + \Delta Z) + z} = \frac{\Delta Z}{2z}
\]

In the limit as \(\Delta z \to 0\), we have an exact differential:

\[
dr = \frac{\Delta r}{\Delta z} dz = \frac{1}{2} \frac{dz}{2z} \\
\frac{dln f(z)}{dz} = \frac{1}{2} \frac{df(z)}{dz}
\]

Then, by using the theory of small reflections, the total reflection coefficient at \(z = 0\)

\[
\frac{\Delta r}{\Delta z} = \frac{2z}{2z + \Delta z}
\]

FIGURE 6.18  A tapered transmission line matching section and the model for an incremental length of tapered line. (a) The tapered transmission line matching section. (b) Model for an incremental step change in impedance of the tapered line.
can be found by summing all the partial reflections with their appropriate phase shifts:

\[ \Gamma(\theta) = \frac{1}{2} \int_{-\theta}^{\theta} e^{-i2\beta z} \frac{d}{dz} \left( \frac{Z(z)}{Z_0} \right) dz \]

where \( \theta = 2\beta z \). So if \( Z(z) \) is known, \( \Gamma(\theta) \) can be found as a function of frequency. Alternatively, if \( \Gamma(\theta) \) is specified, then in principle \( Z(z) \) can be found. This latter procedure is difficult, and is generally avoided in practice; the reader is referred to references [1], [4] for further discussion along these lines. Here we will consider three special cases of \( Z(z) \) impedance tapers, and evaluate the resulting responses.

**Exponential Taper**

Consider first an exponential taper, where

\[ Z(z) = Z_0 e^{a z}, \quad \text{for } 0 < z < L, \]

as indicated in Figure 6.19a. At \( z = 0 \), \( Z(0) = Z_0 \), as desired. At \( z = L \), we wish to have \( Z(L) = Z_L = Z_0 e^{aL} \), which determines the constant \( a \) as

\[ a = \frac{1}{L} \ln \left( \frac{Z_L}{Z_0} \right). \]

We now find \( \Gamma(\theta) \) by using (6.68) and (6.69) in (6.67):

\[ \Gamma = \frac{1}{2} \int_{-\theta}^{\theta} e^{-i2\beta z} \frac{d}{dz} \left( \ln e^{a z} \right) dz = \frac{1}{2L} \int_{-\theta}^{\theta} e^{-i2\beta z} \frac{d}{dz} \ln Z_0 \frac{Z_0}{Z_L} \frac{Z_L}{Z_0} \frac{Z_L}{Z_L - e^{-i2\beta z}} \ln \beta L \frac{Z_L}{Z_L} \frac{Z_L}{Z_L - e^{-i2\beta z}} = \frac{1}{2L} \ln \beta L. \]

Observe that this derivation assumes that \( \beta \), the propagation constant of the tapered line, is not a function of \( z \); an assumption which is generally valid only for TEM lines.

The magnitude of the reflection coefficient in (6.70) is sketched in Figure 6.19b; note that the peaks in \( |\Gamma(\theta)| \) decrease with increasing length, as one might expect, and that the length should be greater than \( \lambda/2 \) (\( \beta L > \lambda \)) to minimize the mismatch at low frequencies.

**Triangular Taper**

Next consider a triangular taper for \( \frac{d}{dz} \ln \frac{Z(z)}{Z_0} \), that is,

\[ Z(z) = \begin{cases} Z_0 e^{z/L} & \text{for } 0 < z < L/2 \\ Z_0 e^{(2L-z)/L} & \text{for } L/2 < z < L. \end{cases} \]

Then,

\[ \frac{d}{dz} \ln \frac{Z(z)}{Z_0} = \begin{cases} \frac{1}{4z} & \text{for } 0 < z \leq L/2 \\ \frac{1}{4(L - z)} & \text{for } L/2 < z \leq L. \end{cases} \]

**Klopfenstein Taper**

Considering the fact that there are an infinite number of possibilities for choosing an impedance matching taper, it is logical to ask if there is a design which is "best." For a given taper length (greater than a critical value), the Klopfenstein impedance taper [4], [5] has been shown to be optimum in the sense that the reflection coefficient is minimum over the passband. Alternatively, for a maximum reflection coefficient specification in the passband, the Klopfenstein taper yields the shortest matching section.

The Klopfenstein taper is derived from a stepped Chebyshev transform as the number of sections increases to infinity, and is analogous to the Taylor distribution of antenna array theory. We will not present the details of this derivation, which can be found in references [1], [4], only the necessary results for the design of Klopfenstein tapers are given below.
The logarithm of the characteristic impedance variation for the Klopfenstein taper is given by
\[
\ln Z(z) = \frac{1}{2} \ln Z_L Z_0 + \frac{\Gamma_0}{\cosh A} A^2 (2z/L - 1, A), \quad 0 \leq z \leq L.
\]  
6.74

where the function \( \phi(x, A) \) is defined as
\[
\phi(x, A) = -\phi(-x, A) = \int_0^x \frac{I_1(A \sqrt{1 - y^2})}{A \sqrt{1 - y^2}} \, dy, \quad |x| \leq 1.
\]  
6.75

where \( I_1(x) \) is the modified Bessel function. This function takes the following special values:
\[
\begin{align*}
\phi(0, A) &= 0 \\
\phi(x, 0) &= \frac{x}{2} \\
\phi(1, A) &= \frac{\cosh A - 1}{A},
\end{align*}
\]

but otherwise must be calculated numerically. A very simple and efficient method for doing this is available [6].
Chapter 6: Impedance Matching and Tuning

and (6.78) gives $A$ as

$$A = \cosh^{-1} \left( \frac{\Gamma_L}{\Gamma_m} \right) = \cosh^{-1} \left( \frac{0.346}{0.002} \right) = 3.543.$$ 

The impedance taper must be numerically evaluated from (6.74). The reflection coefficient magnitude is given by (6.76):

$$|\Gamma(\theta)| = \Gamma_L \cosh^{\sqrt{BL - A}} \cosh A.$$

The passband for the Klopfenstein taper is defined as $BL > A = 3.543 = 1.13\pi$.

---

**Figure 6.21** Solution to Example 6.9. (a) Impedance variations for the triangular, exponential, and Klopfenstein tapers. (b) Resulting reflection coefficient magnitude versus frequency for the tapers of (a).

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6.9 The Bode-Fano Criteria

Figure 6.21a,b shows the impedance variations (versus $z/L$), and the resulting reflection coefficient magnitude (versus $BL$) for the three types of tapers. The Klopfenstein taper is seen to give the desired response of $|\Gamma| \leq \Gamma_m = 0.02$ for $BL \geq 1.13\pi$, which is lower than either the triangular or exponential taper responses. Also note that, like the stepped-Chaplygin matching transformer, the response of the Klopfenstein taper has equal-ripple lobes versus frequency in its passband.

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**THE BODE-FANO CRITERIA**

In this chapter we discussed several techniques for matching an arbitrary load at a single frequency, using lumped elements, tuning stubs, and single-section quarter-wave transformers. We then presented multisection transformers and tapered lines as a means of obtaining broader bandwidths, with various passband characteristics. We will now close our study of impedance matching with a somewhat qualitative discussion of the theoretical limits that constrain the performance of an impedance matching network.

We limit our discussion to the circuit of Figure 6.1, where a lossless network is used to match an arbitrary complex load, generally over a nonzero bandwidth. From a very general perspective, we might raise the following questions in regard to this problem:

- Can we achieve a perfect match (zero reflection) over a specified bandwidth?
- If not, how good can we do? What is the trade-off between $\Gamma_m$, the maximum allowable reflection in the passband, and the bandwidth?
- How complex must the matching network be for a given specification?

These questions can be answered by the Bode-Fano criteria [7, 8] which gives, for certain canonical types of load impedances, a theoretical limit on the minimum reflection coefficient magnitude that can be obtained with an arbitrary matching network. The Bode-Fano criteria thus represents the optimum result that can be ideally achieved, even though such a result may only be approximated in practice. Such optimal results are always important, however, because they give us the upper limit of performance, and provide a benchmark against which a practical design can be compared.

Figure 6.22a shows a lossless network used to match a parallel RC load impedance. The Bode-Fano criteria states that

$$\int_0^\infty \ln \left( \frac{1}{|\Gamma(\omega)|} \right) d\omega \leq \frac{\pi}{RC}, \quad 6.79$$

where $\Gamma(\omega)$ is the reflection coefficient seen looking into the arbitrary lossless matching network. The derivation of this result is beyond the scope of this text (the interested reader is referred to references [7] and [8]), but our goal here is to discuss the implications of the above result.

Assume that we desire to synthesize a matching network with a reflection coefficient response like that shown in Figure 6.23a. Applying (6.79) to this function gives

$$\int_0^\infty \ln \left( \frac{1}{|\Gamma(\omega)|} \right) d\omega = \int_0^\infty \ln \left( \frac{1}{\Gamma_m} \right) d\omega = \Delta\omega \ln \Gamma_m \leq \frac{\pi}{RC}, \quad 6.80$$
which leads to the following conclusions:

- For a given load (fixed RC product), a broader bandwidth (Δω) can only be achieved at the expense of a higher reflection coefficient in the passband (Γcat).
- The passband reflection coefficient Γcat cannot be zero unless Δω = 0. Thus a perfect match can only be achieved at a finite number of frequencies, as illustrated in Figure 6.23a.
- As R and/or C increase, the quality of the match (Δω and/or 1/Γcat) must decrease. Thus, higher-Q circuits are intrinsically harder to match than are lower-Q circuits.

Since ln |Γ| is proportional to the return loss (in dB) at the input of the matching network, (6.79) can be interpreted as requiring that the area between the return loss curve and the |Γ| = 1 (RL = 0 dB) axis must be less than or equal to a constant. Optimization then implies that the return loss curve should be adjusted so that |Γ| = Γmax over the passband and |Γ| = 1 elsewhere, as shown in Figure 6.23a. In this way, no area under the return loss curve is wasted outside the passband, or lost in regions within the passband for which |Γ| < Γmax.

The square-shaped response of Figure 6.23a is thus the optimum response, but cannot be realized in practice because it would require an infinite number of elements in the matching network. It can be approximated, however, with a reasonably small number of elements, as described in reference [6]. Finally, note that the Chebyshev matching transformer can be considered as a close approximation to the ideal gamut of Figure 6.23a, when the ripple of the Chebyshev response is made equal to Γπ. Figure 6.22 lists the Bode-Fano limits for other types of RC and RL loads.

REFERENCES

LECTURE 5

Smith Chart

To: [Redacted]
From: [Redacted]
Subject: [Redacted]

Reflect. Coefficient: \( \Gamma = |\Gamma| e^{j\phi} = \Gamma_r + j\Gamma_i \)

where: \( \Gamma_r = |\Gamma| \cos \phi \), \( \Gamma_i = |\Gamma| \sin \phi \)

The Smith chart lies in the complex plane of \( \Gamma \).

Example: \( \Gamma_n = 0.3 + j0.4 \)

\[ |\Gamma_1| = \sqrt{0.3^2 + 0.4^2} = 0.5 \]

\[ \phi = \tan^{-1}(0.4/0.3) = 53^\circ \]

\( \Gamma_1 = \frac{z_1 - z_0}{z_1 + z_0} = \frac{2l - zo}{2l + zo} \]

\( \approx \frac{2l - 1}{2l + 1} \)

\( \approx \frac{1 + l}{1 - l} \)

When both \( \Gamma_r \) and \( \Gamma_i \) are negative numbers, \( \phi \) is in the third quadrant in the \( \Gamma_r - \Gamma_i \) plane.

The unit circle corresponds to \( |\Gamma| = 1 \). Because \( |\Gamma| \leq 1 \) for a trans line, only that part of the \( \Gamma_r - \Gamma_i \) plane that lies within the unit circle has physical meaning.

Impedances on a Smith chart are represented by normalized values, with \( z_0 \), the characteristic impedance of the line, serving as the normalization constant.

\[ z_0 = \frac{z}{z_0} \]
Ref. Sec. 3.2

\[
\Gamma = \frac{Z_0 - Z_L}{Z_0 + Z_L} = \frac{Z_L - 1}{Z_0} = \frac{\bar{Z}_L - 1}{\bar{Z}_L + 1}
\]

\[
\bar{Z}_L = \frac{1 + \Gamma}{1 - \Gamma}
\]

Expressing:  \( \bar{Z}_L = r_L + j\chi_L \),  \( \Gamma = \sigma + j\pi \)

\[
r_L = \frac{1 - \Gamma^2 - \pi^2}{(1 - \sigma)^2 + \pi^2} \quad \chi_L = \frac{2\pi}{(1 - \sigma)^2 + \pi^2}
\]

\((r_L, \chi_L)\) unique \(\leftrightarrow\) \((\sigma, \pi)\)

If we fix \( r_L \), many possible combinations of values can be assigned to \( \sigma, \pi \), each of which give the same value of \( r_L \) and all of them are on a constant \( r_L \) circle. (for \( r_L = 0 \), \( |\Gamma| = 1 \), cross \((\sigma, \pi) = (1,0)\))

![Constant r_L circles](image)

![Constant \chi_L circles](image)

A given point on the Smith chart, such as \( \bar{Z}_L = 2 - j4 \) represents an equivalent reflection coefficient \( \Gamma \) values 0.45 \((-26.6^\circ)\). The magnitude \( |\Gamma| = 0.45 \) is obtained by dividing the length of the line between the center of the Smith chart and the point \( P \) by the length of
of the line between the center of the Smith chart and the edge of the unit circle (this radius of the unit circle corresponds to \( r_1 = 1 \)).

The innermost scale of the perimeter of the Smith chart is labeled angle of reflection coefficient in degrees.

\[
Z_{in} = Z_0 \left[ \frac{1 + R e^{-j2\beta e}}{1 - R e^{-j2\beta e}} \right]
\]

Normalized
\[
\frac{Z_{in}}{Z_0} = \frac{Z_{in}}{Z_0} = \frac{1 + R e^{-j2\beta e}}{1 - R e^{-j2\beta e}}
\]

\( \Gamma = 11 e^{j\beta r} \) is the voltage reflection coefficient at the load.

\( \Gamma_c = R e^{-j2\beta e} = 11 e^{j(\beta r - 2\beta e)} \) is the phase-shifted voltage reflection coefficient.

Same magnitude as \( \Gamma \), but the phase is shifted by \( 2\beta e \) relative to that of \( \Gamma \).

\[
( \frac{Z_{in}}{Z_0} = \frac{1 + \Gamma_c}{1 - \Gamma_c} )
\]

To transform \( \Gamma \rightarrow \Gamma_c \) (or \( Z_{in} \rightarrow \tilde{Z}_{in} \))

1. Identify position of \( Z_c \) on Smith chart (Point A)
2. Draw a circle through A with the center of the circle being at the center of the Smith chart (constant \( r_1 \))
3. Move clockwise on the circle adding to the phase of point A (intersection of radius to A and outer circle — outermost label (waveguide toward generator))
the distance of the position of \( z \).

(4) Denote \( z_2 \) by multiplying with \( z_0 \).

(Distance larger than 0.5\( \lambda \) or smaller than 0 have to be shrunk to this interval by adding / subtracting half wavelengths)

\[ z_2 = (0.5 - j0.5) \, R \quad z_0 = 10 \, R \]

\[ z_1 = z_2 / z_0 = 2 - j1 \quad (0.287, \text{ on the scale}) \]

\( \hat{z}_1 \) (\( z = 0.12 \)) \( \rightarrow \) Add 0.287 + 0.1 = 0.387

\( \rightarrow \) New point at the constant \( R \) circle

\[ \hat{z}_1 = 0.6 - j0.6 \quad \rightarrow \hat{z}_2 = \hat{z}_1 - z_0 = (30 - 10) \, \text{j32} \]

---

**Standing Wave Ratio**

The constant - \( R \) circle intersects the real axis \( (\Re) \) at two points, \( P_{\text{Max}} \) and \( P_{\text{Min}} \). At both points \( R = 0 \), \( R = R_\text{r} \), \( x = 0 \).

From definition of \( R \):

\[ R = \frac{\hat{z}_l - 1}{\hat{z}_l + 1} \]

\( P_{\text{Min}}, P_{\text{Max}} \) correspond to:

\[ R = R_\text{r} = \frac{\hat{z}_l - 1}{\hat{z}_l + 1} \quad (\text{for } \Re \neq 0) \]

(P\text{Min when } \Re < 1, \text{ P\text{Max when } } \Re > 1)

\[ |R| = \frac{S - 1}{S + 1} \quad S = 5 \, \text{W} \quad \text{For } \text{Max}, \text{Min } \rightarrow |R| = R \]

\[ R = \frac{S - 1}{S + 1} \]
and

\[ \theta_t = \tan^{-1}(0.4/0.3) = 53^\circ. \]

Similarly, point \( B \) represents \( \Gamma_B = -0.5 - j0.2 \), or \( |\Gamma_B| = 0.54 \) and \( \theta_t = 202^\circ \) [or, equivalently, \( \theta_t = (360^\circ - 202^\circ) = -158^\circ \)]. Note that when both \( \Gamma_t \) and \( \Gamma_1 \) are negative numbers \( \theta_t \) is in the third quadrant in the \( \Gamma_t-\Gamma_1 \) plane. Thus, when using \( \theta = \tan^{-1}(\Gamma_1/\Gamma_t) \) to compute \( \theta_t \), it may be necessary to add or subtract 180° to obtain the correct value of \( \theta_t \).

The unit circle shown in Fig. 2-20 corresponds to \( |\Gamma| = 1 \). Because \( |\Gamma| \leq 1 \) for a transmission line, only that part of the \( \Gamma_t-\Gamma_1 \) plane that lies within the unit circle has physical meaning; hence, future drawings will be limited to the domain contained within the unit circle.

Impedances on a Smith chart are represented by
meaningless). Hence, Eq. (2.100) can generate two families of circles, one family corresponding to positive values of \( x_L \) and another corresponding to negative values of \( x_L \). Furthermore, as shown in Fig. 2.21, only part of a given circle falls within the bounds of the unit circle. The families of circles of the two parametric equations given by Eqs. (2.98) and (2.100) plotted for selected values of \( r_L \) and \( x_L \) constitute the Smith chart shown in Fig. 2.22. A given point on the Smith chart, such as point \( P \) in Fig. 2.22, represents a normalized load impedance \( z_L = 2 - j1 \), with a corresponding voltage reflection coefficient \( \Gamma = 0.45 \exp(-j26.6^\circ) \). The magnitude \( |\Gamma| = 0.45 \) is obtained by dividing the length of the line between the center of the Smith chart and the point \( P \) by
the length of the line between the center of the Smith chart and the edge of the unit circle (the radius of the unit circle corresponds to $|\Gamma| = 1$). The perimeter of the Smith chart contains three concentric scales. The innermost scale is labeled angle of reflection coefficient in degrees. This is the scale for $\theta_r$. As indicated in Fig. 2-22, $\theta_r = -26.6^\circ$ for point $P$. The meanings and uses of the other two scales are discussed next.
voltage standing-wave ratio (SWR) is related to \(|\Gamma|\) by Eq. (2.59) as

\[
S = \frac{1 + |\Gamma|}{1 - |\Gamma|}.
\]

(2.107)

Thus, a constant value of \(|\Gamma|\) corresponds to a specific value for \(S\). As was stated earlier, to transform \(z_L\) to \(z_{in}\), we need to maintain \(|\Gamma|\) constant, which means staying on the SWR circle, and to decrease the phase of \(\Gamma\) by \(2\pi l\). This is equivalent to moving a distance \(l = 0.1\lambda\) toward

Figure 2-23: Point A represents a normalized load \(z_L = 2 - j1\) at 0.287\(\lambda\) on the WTG scale. Point B represents the line input at 0.1A from the load. At \(B\), \(z_{in} = 0.6 - j0.66\).
2-9.4 Impedance to Admittance Transformations

In solving certain types of transmission line problems, it is often more convenient to work with admittances than with impedances. Any impedance $Z$ is in general a complex quantity consisting of a resistance $R$ and a reactance $X$:

$$Z = R + jX \quad (\Omega) \quad (2.112)$$
Solution: (a) The normalized load impedance is

\[ z_L = \frac{Z_L}{Z_0} = \frac{25 + j50}{50} = 0.5 + j1, \]

which is marked as point A on the Smith chart in Fig. 2.26. Using a ruler, a radial line is drawn from the center of the chart at point O through point A, outward to the outer perimeter of the chart. The line crosses the scale labeled “angle of reflection coefficient in degrees” at \( \theta_r = 83^\circ \). Next, a ruler is used to measure the length \( d_A \)
of the line between points $O$ and $A$ and the length $d_{O'}$ of the line between points $O$ and $O'$, where $O'$ is an arbitrary point on the $r_{L0} = 0$ circle. The length $d_{O'}$ is equal to the radius of the $|\Gamma| = 1$ circle. The magnitude of $\Gamma$ is then obtained from $|\Gamma| = \frac{d_A}{d_{O'}} = 0.62$. Hence,

$$\Gamma = 0.62e^{j39^\circ} = 0.62/e^{39^\circ}.$$  \hfill (2.121)

(b) Using a compass, the SWR circle with center at
Q2.21 What line length corresponds to one complete rotation around the Smith chart? Why?

Q2.22 What points on the SWR circle correspond to the locations of the voltage maxima and minima on the line and why?

Q2.23 Given a normalized impedance \( z_L \), how do you use the Smith chart to find the corresponding normalized admittance \( y_L = 1/z_L \)?