Wed.
4:30
Antennas
VL 341

~~HW due Thurs.~~
4

EE 3065

Georgia Institute of Technology
Atlanta, Georgia 30332

~~thr. → No fiber optics/ oblique~~
~~No Smith chart incidence~~
~~No TE/TM polarization~~

MEMORANDUM

~~Test 2 HW 4,5,6~~
~~covers up to cavity~~
~~dielectric ‖ and + putor~~
~~parallel waveguides~~
~~rectangular~~
~~no ridge/ surface waveguides~~

~~Date:~~ 2.1

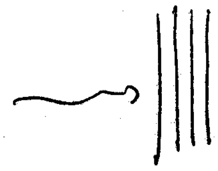
Final - mostly last 3rd of class
hw's, sample final

LECTURE 2

~~To:~~
~~From:~~
~~Subject:~~ Review of Wave Equation

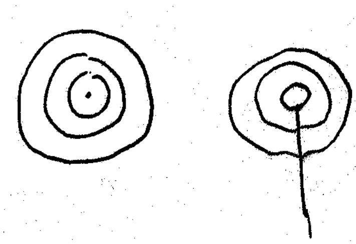
Moving waves carry energy from one point to another.
Waves have velocity (Light waves in vacuum propagate with 3×10^8 m/sec; sound waves in air 330 m/sec)
Some waves (Most EM and sound waves) exhibit linearity
~ the total of two linear waves is the sum of the two waves as they would exist separately.

(·) Plane waves: characterized by a disturbance that at a given (free space) point of time has uniform properties across an infinite plane perpendicular to the direction of propagation



(·) cylindrical waves : — uniform across cylindrical surface

(·) spherical waves : — uniform across spherical surface



A medium is lossless if it doesn't attenuate the amplitude of the wave traveling within it or on its surface.

$$y(x,t) = \underbrace{A}_{\text{amplitude}} \cos\left(\frac{2\pi t}{\underbrace{T}_{\text{time period}}} - \frac{2\pi x}{\underbrace{\lambda}_{\text{spatial wavelength}}} + \underbrace{\varphi_0}_{\text{reference phase}}\right) \quad (*)$$

Phase equivalent expression:

$$y(x,t) = A\cos\varphi(x,t)$$

$$\varphi(x,t) = \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \varphi_0\right) \rightsquigarrow \text{instantaneous phase (rads)}$$

2π rads = $360°$

The wave pattern repeats itself at a spatial period λ along x and at a temporal period T along t.

The phase velocity: $u_p = \frac{dx}{dt} = \frac{\lambda}{T}$ (m/s) (= propagation velo

is the velocity of the wave pattern as it moves.

If one of the signs is positive and the other is negative, then → positive x-direction; if both are positive or ne then → negative x-direction.

Frequency: $f = 1/T$ (Hz) $\rightsquigarrow u_p = f\cdot\lambda$ (m/s)

↳ $y(x,t) = A\cos\left(2\pi f t - \frac{2\pi}{\lambda}x\right)$

Angular velocity: $\omega = 2\pi f$ (rad/s)

Phase constant (wavenumber) $\beta = \frac{2\pi}{\lambda}$ (rad/m) $\Big\}$ $y(x,t) = A\cos(\omega t -$

(Propagation to $-x$-direction) $\Rightarrow y(x,t) = A\cos(\omega t + \beta x)$

Lossy Medium: $y(x,t) = Ae^{-\alpha x}\cos(\omega t - \beta x + \varphi_0)$

α: attenuation factor (Np/m)

Complex Numbers

$Z = x + jy$, $x = Re(z)$, $y = Im(z)$

Polar form: $z = |z| e^{j\vartheta} = |z| \angle \vartheta$

Magnitude Angle

Euler's Identity: $e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$

$\left(\begin{array}{l} x = |z| \cos\vartheta, \quad y = |z| \sin\vartheta \\ |z| = \sqrt{x^2 + y^2}, \quad \vartheta = \tan^{-1}(y/x) \end{array} \right)$

Complex conjugate: $z^* = (x+jy)^* = x - jy = |z| \angle -\vartheta$

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

$z_1 \cdot z_2 = |z_1||z_2| e^{j(\vartheta_1 + \vartheta_2)}$, $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\vartheta_1 - \vartheta_2)}$

$z^n = |z|^n e^{j\vartheta n} = |z|^n (\cos(n\vartheta) + j\sin(n\vartheta))$

$z^{1/2} = |z|^{1/2} e^{j\vartheta/2}$

$-1 = e^{j\pi} = 1 \angle \pi$

$j = e^{j\pi/2} = 1 \angle \pi/2$

$-j = e^{-j\pi/2} = 1 \angle -\pi/2$

$\sin x = \cos(x - \pi/2)$

$\sin(-x) = -\sin(x)$

Phasors are useful for single-frequency waves.

(Δ Adopt a cosine reference) $(\sin x = \cos(\frac{\pi}{2} - x), \; \cos(-x) = \cos x)$

$v_s(t) = V_0 \sin(\omega t + \varphi_0) = V_0 \cos(\frac{\pi}{2} - \omega t - \varphi_0) = V_0 \cos(\omega t + \varphi_0 - \frac{\pi}{2})$

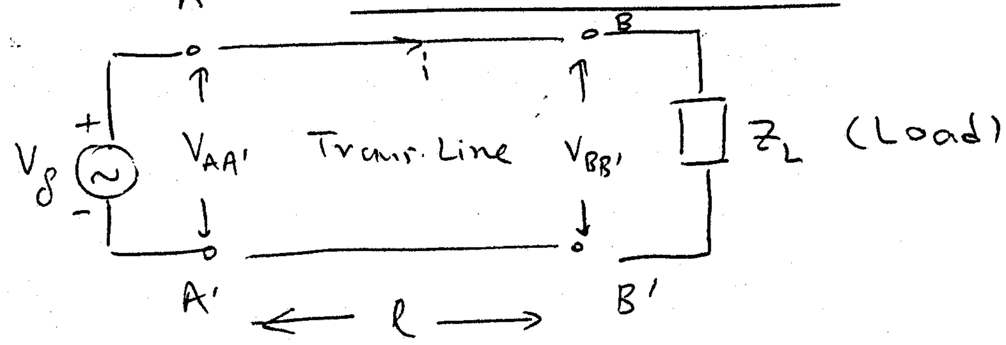
$z(t) = Re[\tilde{Z} e^{j\omega t}] \Rightarrow v_s(t) = Re[V_0 e^{j(\omega t + \varphi_0 - \frac{\pi}{2})}] =$

phasor $= Re[V_0 e^{j(\varphi_0 - \pi/2)} e^{j\omega t}] = Re[\tilde{V}_s e^{j\omega t}]$

$\tilde{V}_s = V_0 e^{j(\varphi_0 - \pi/2)}$

~~LECTURE~~ 2

Transmission Lines



$c = \lambda f$

$V_{AA'} = V_g(t) = V_0 \cos(\omega t)$ (V)

$V_{BB'} = V_{AA'}(t - l/c) = V_0 \cos[\omega(t - l/c)]$

Phase Difference: $\frac{\omega l}{c} = \frac{2\pi f l}{c} = \boxed{2\pi \frac{l}{\lambda}}$ rads

If $l/\lambda < 0.01$ → tr. line effects can be ignored $l \lll \lambda$ low f

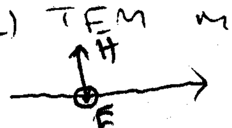
If $l/\lambda \gtrsim 0.01$ → consider tr. line effects

+reflections, power loss, dispersion

Propagation Modes

$\lambda = \frac{V_p}{f}$

(1) TEM modes: Electric and Magnetic fields are entirely transverse to the direction of propagation

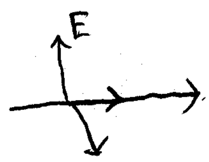


can use circuit theory.

(Quasi-TEM modes: nontransverse field components are << transver

TEM lines: consist of two separate conducting surfaces

(2) Higher-order modes: At least one significant field component in the direction of propagation

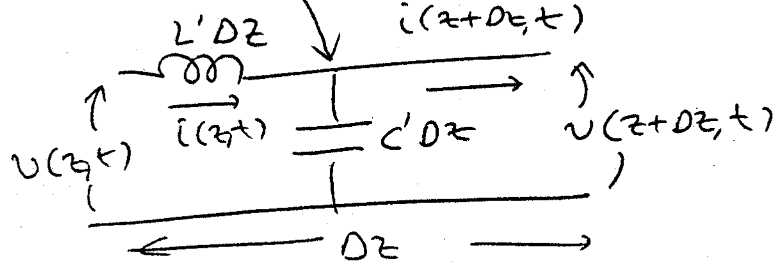


use Maxwell's eq.

Parallel-wire configuration regardless of the specific shape of the line under consideration (Applicable to TEM

(A) Lossless (TEM line)

L' : combined inductance of both conductors per unit length (
C' : capacitance of two conductors per unit length (F/m)



$$-\frac{\partial v(z,t)}{\partial z} = L' \frac{\partial i(z,t)}{\partial t}, \quad -\frac{\partial i(z,t)}{\partial z} = C' \frac{\partial v(z,t)}{\partial t}$$

→ Telegrapher's equations

Phasors: $v(z,t) = Re[\tilde{V}(z) e^{j\omega t}]$, $i(z,t) = Re[\tilde{I}(z) e^{j\omega t}]$

$$-\frac{d\tilde{V}(z)}{dz} = j\omega L' \tilde{I}(z), \quad -\frac{d\tilde{I}(z)}{dz} = j\omega C' \tilde{V}(z)$$

$$\frac{d^2\tilde{V}(z)}{dz^2} - j\omega L' j\omega C' \tilde{V}(z) = 0 \qquad (// \tilde{I}(z))$$

$$\Rightarrow \frac{d^2\tilde{V}(z)}{dz^2} + \beta^2 \tilde{V}(z) = 0 \Rightarrow \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j}$$

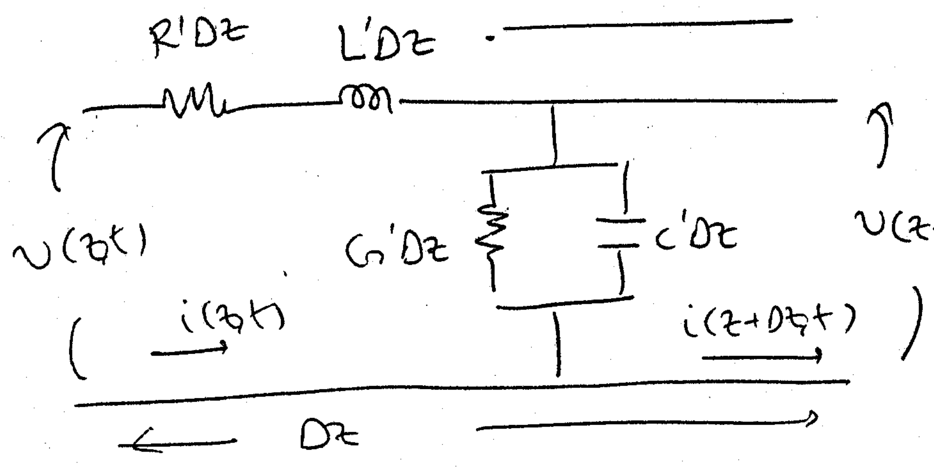
β : phase constant, $\beta = \omega\sqrt{L'C'}$, $u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$

For TEM lines, $\beta = \omega\sqrt{\mu\varepsilon}$ (rad/m), $u_p = \frac{1}{\sqrt{\mu\varepsilon}}$ (m/sec)

$\varepsilon = \varepsilon_r \varepsilon_0 \Rightarrow u_p = \frac{c_0}{\sqrt{\varepsilon_r}}$, $c_0 = 3\times10^8$ m/sec $\not\propto f$ ⇒ Nondispersive all TEM

$$\lambda = \frac{u_p}{f} = \frac{c_0}{f\sqrt{\varepsilon_r}}$$

Lossy Line



$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$

R': combined Resistance of both conductors per unit length (

G': conductance of insulation medium per unit length (S/m)

$$-\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t}, \quad -\frac{\partial i(z,t)}{\partial z} = G' v(z,t) + C' \frac{\partial v(z,t)}{\partial t}$$

Phasors: $-\frac{d\tilde{V}(z)}{\partial z} = (R' + j\omega L')\tilde{I}(z), \quad -\frac{d\tilde{I}(z)}{\partial z} = (G' + j\omega C')\tilde{V}$

&

$$\frac{d^2\tilde{V}(z)}{dz^2} - (R' + j\omega L')(G' + j\omega C')\tilde{V}(z) = 0$$

$$\Leftrightarrow \frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2 \tilde{V}(z) = 0 \qquad \text{(// } \tilde{I}(z)\text{)}$$

Complex propagation constant: $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$

$= a + j\beta$

atten. constant ≠ lossless

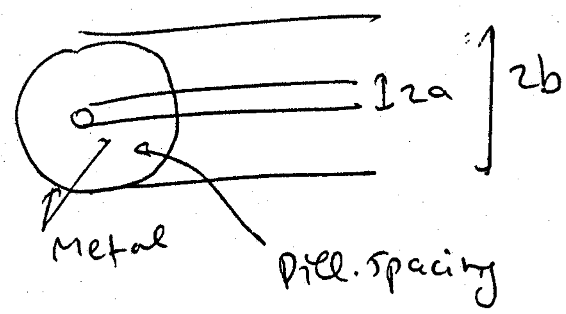
$$a = Re(\gamma), \quad \beta = Im(\gamma), \quad u_p = \frac{\omega}{\beta}, \quad \lambda = \frac{u_p}{f}$$

<u>For TEM lines</u>: $L'C' = \mu\varepsilon, \quad \frac{G'}{C'} = \frac{\sigma}{\varepsilon}$

$\frac{R'}{L'} = \frac{G'}{C'}$ TEM condition

Example 1

Coax Line:



$$L' = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)}$$

Ridge coaxial Air line with inner conductor diameter of 0.6 cm and outer conductor diameter of 1.2 cm. Conductors made by copper

$\varepsilon = \varepsilon_0 \varepsilon_r$, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\varepsilon_r \cong 1$

Conductivity: $\sigma = 5.8 \times 10^{7}$ S/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m

$$L' = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) = 0.14 \ (\mu H/m)$$

$$C' = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)} = 80.3 \ (pF/m)$$

surface resistance of metal

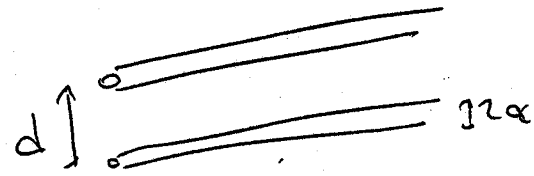
$$R' = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right), \quad R_s = \sqrt{\pi f \mu_c / \sigma_c} \Rightarrow R' = 2.08\times 10 \ \Omega/m$$

conductors (usu. $\mu_0 = \mu_c$)

$$G' = \frac{2\pi\sigma}{\ln(b/a)} \overset{\sigma_{air}=0}{=\!=} 0 \ S/m$$

Example 2

Two wire line



Separated by distance 2 cm and each is 1 mm in radius. The wires may be treated as perfect conductors with $\sigma_c = \infty$.

$$R' = \frac{R_s}{\pi a} = \frac{\sqrt{\pi f \mu_c / \sigma_c}}{\pi a} \overset{\sigma_c \to \infty}{=} 0$$

$$L' = \frac{\mu}{\pi} \ln\left[(d/2a) + \sqrt{(d/2a)^2 - 1} \right] \overset{\mu = \mu_0}{=\!=} 1.20\ \mu H/$$

$$G' = \frac{\pi \sigma}{\ln\left[(d/2a) + \sqrt{(d/2a)^2 - 1}\right]} \overset{\sigma_{air} \approx 0}{=\!=} 0$$

$$C' = \frac{\pi \varepsilon}{\ln\left[(d/2a) + \sqrt{(d/2a)^2 - 1}\right]} \overset{\varepsilon = \varepsilon_0}{=\!=} 9.29\ pF/m$$

$(d = 2cm,\ a = 1mm)$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

2-1 General Considerations

In most electrical engineering curricula, the study of electromagnetics is preceded by one or more courses on electrical circuits. In this book, we use this background to build a bridge between circuit theory and electromagnetic theory. The bridge is provided by transmission lines, the topic of this chapter. By modeling the transmission line in the form of an equivalent circuit, we can use Kirchhoff's voltage and current laws to develop wave equations whose solutions provide an understanding of wave propagation, standing waves, and power transfer. Familiarity with these concepts facilitates the presentation of material in later chapters.

Although the family of ***transmission lines*** may encompass all structures and media that serve to transfer energy or information between two points, including nerve fibers in the human body, acoustic waves in fluids, and mechanical pressure waves in solids, we shall focus our treatment in this chapter on transmission lines used for guiding electromagnetic signals. Such transmission lines include telephone wires, coaxial cables carrying audio and video information to TV sets or digital data to computer monitors, and optical fibers carrying light waves for the transmission of data at very high rates. Fundamentally, a transmission line is a two-port network, with each port consisting of two terminals, as illustrated in Fig. 2-1. One of the ports is the sending end and the other is the receiving end. The source connected to its sending end may be any circuit with an output voltage, such as a radar transmitter, an amplifier, or a computer terminal operating in the transmission mode. From circuit theory, any such source can be represented by a Thévenin-equivalent ***generator circuit*** consisting of a generator voltage V_g in series with a generator resistance R_g, as shown in Fig. 2-1. The generator voltage may consist of digital pulses, a modulated time-varying sinusoidal signal, or any other signal waveform. In the case of a-c signals, the generator circuit is represented by a voltage phasor $\widetilde{V}_g$ and an impedance Z_g.

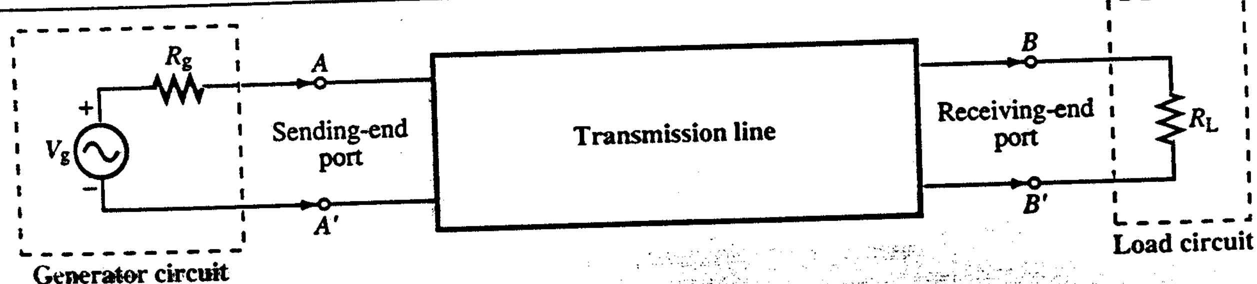


Figure 2-1: A transmission line is a two-port network connecting a generator circuit at the sending end to a load at the receiving end.

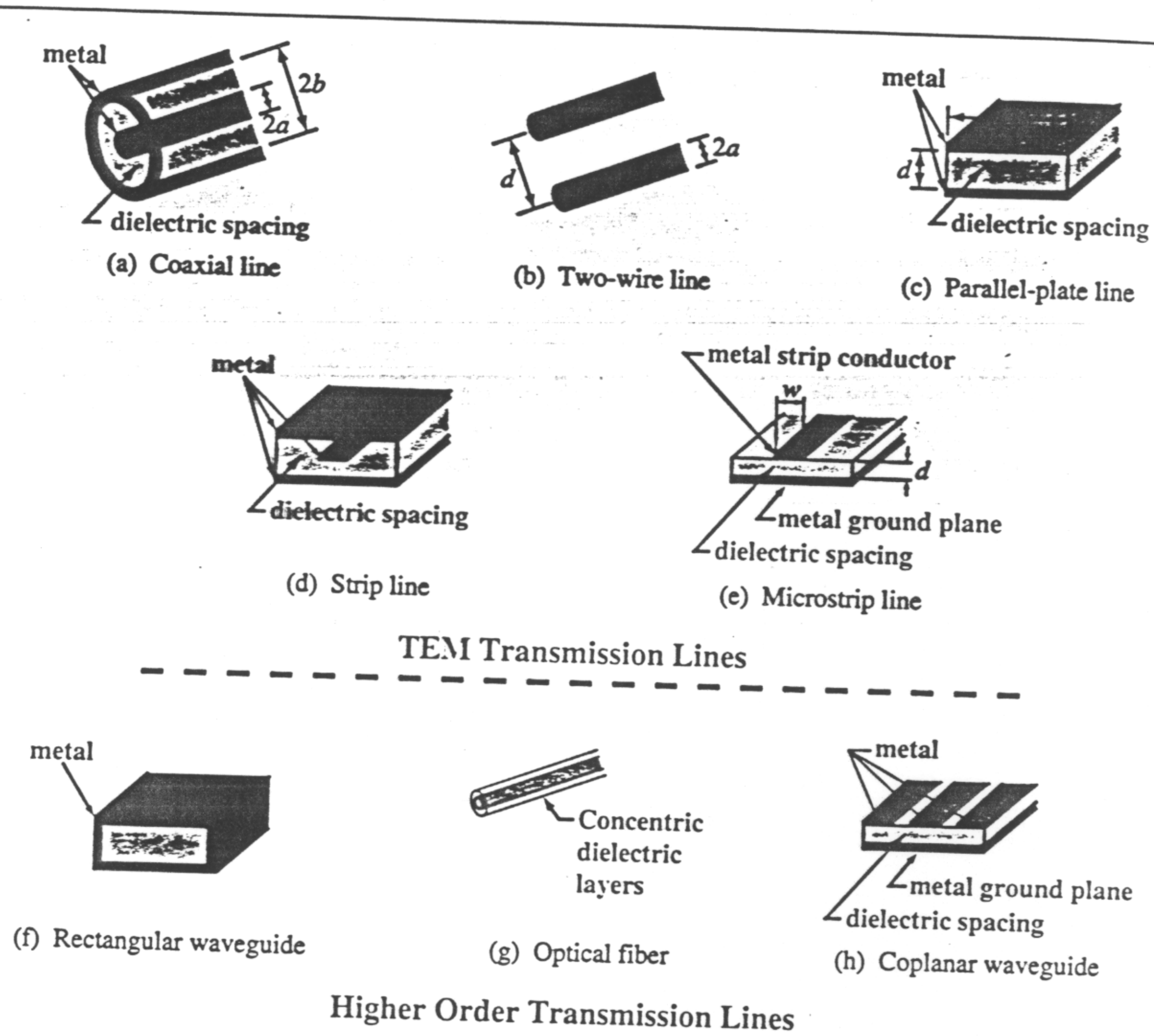


Figure 2-4: A few examples of transverse electromagnetic (TEM) and higher-order transmission lines.

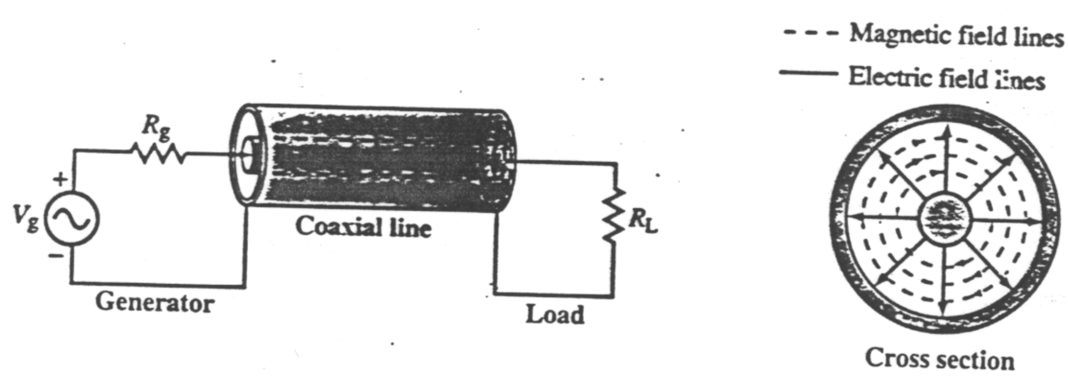


Figure 2-5: In a coaxial line, the electric field lines are in the radial direction between the inner and outer conductors, and the magnetic field forms circles around the inner conductor.

Table 2-1: Transmission-line parameters R', L', G', and C' for three types of lines.

Parameter	Coaxial	Two Wire	Parallel Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_s}{\pi a}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]$	$\frac{\mu d}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]}$	$\frac{\sigma w}{d}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]}$	$\frac{\varepsilon w}{d}$	F/m

Notes: (1) Refer to Fig. 2-4 for definitions of dimensions. (2) μ, ε, and σ pertain to the insulating material between the conductors. (3) $R_s = \sqrt{\pi f \mu_c/\sigma_c}$. (4) μ_c and σ_c pertain to the conductors. (5) If $(d/2a)^2 \gg 1$, then $\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right] \simeq \ln(d/a)$.

Table 2-2: Characteristic parameters of transmission lines.

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General case	$\gamma = \sqrt{(R'+j\omega L')(G'+j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\frac{(R'+j\omega L')}{(G'+j\omega C')}}$
Lossless ($R' = G' = 0$)	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \sqrt{L'/C'}$
Lossless coaxial	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (60/\sqrt{\varepsilon_r})\ln(b/a)$
Lossless two wire	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (120/\sqrt{\varepsilon_r}) \cdot \ln[(d/2a)+\sqrt{(d/2a)^2-1}]$ $Z_0 \simeq (120/\sqrt{\varepsilon_r})\ln(d/a)$, if $d \gg a$
Lossless parallel plate	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (120\pi/\sqrt{\varepsilon_r})(d/w)$

Notes: (1) $\mu = \mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$, $c = 1/\sqrt{\mu_0\varepsilon_0}$, and $\sqrt{\mu_0/\varepsilon_0} \simeq (120\pi)\ \Omega$, where ε_r is the relative permittivity of insulating material. (2) For coaxial line, a and b are radii of inner and outer conductors. (3) For two-wire line, a = wire radius and d = separation between wire centers. (4) For parallel-plate line, w = width of plate and d = separation between the plates.

Georgia Institute of Technology
Atlanta, Georgia 30332

EE3065

MEMORANDUM

1

Date:

To:

From:

Subject:

LECTURE 3

General (lossy) trans. line

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} = \frac{\gamma}{R'+j\omega L'}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$

by replacing $\tilde{V}(z)$ in: $-\frac{d\tilde{I}(z)}{dz} = (G'+j\omega C')\,\tilde{V}(z)$

$$\gamma = \sqrt{(R'+j\omega L')(G'+j\omega C')}$$

Define the characteristic impedance

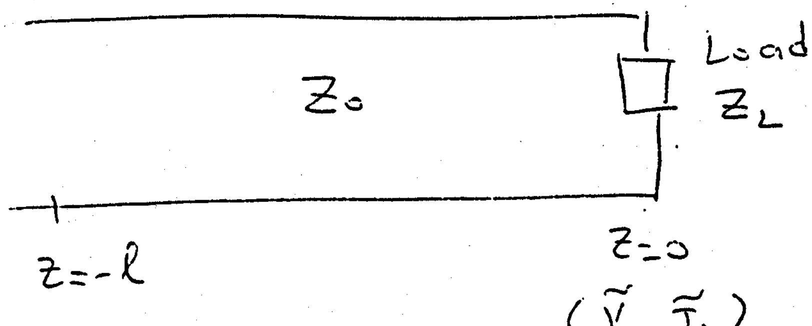
$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R'+j\omega L'}{\gamma} = \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}} \quad \left(\text{Lossless: } Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\mu}{\varepsilon}}\right.$$

→ the ratio of the voltage to the current amplitude for each of the travelling waves individually (with an additional (-) for $\to -\hat{z}$).

IT IS NOT EQUAL TO THE RATIO OF TOTAL VOLTAGE TO THE TOTAL CURRENT, UNLESS ONE OF THE TWO WAVES IS ABSENT!!

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

Voltage Reflection Coefficient



$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L}, \quad \tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-, \quad \tilde{I}_L = \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

$$\rightarrow V_0^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^+ \Rightarrow$$

$$\Rightarrow \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = -\frac{I_0^-}{I_0^+}$$

(·) ALWAYS: $|\Gamma| \le 1$

(·) A load is matched to the line if $Z_L = Z_0 \Rightarrow \Gamma = 0$ (no reflection)

(·) Load: open circuit ($Z_L = \infty$) $\Rightarrow \Gamma = 1$

(·) Load: short circuit ($Z_L = 0$) $\Rightarrow \Gamma = -1$

(·) Load: purely reactive ($Z_L = jX_L$) $\Rightarrow |\Gamma| = 1$, $\Gamma \in \mathbb{C}$

$$\tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{+j\beta z})$$
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z})$$

$\rightarrow$ Max: $V_0^+ (1 + |\Gamma|)$
Min: $V_0^+ (1 - |\Gamma|)$
(sinusoidal wave $\rightarrow$ standing wave)

Define voltage standing wave ratio

$$VSWR = S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

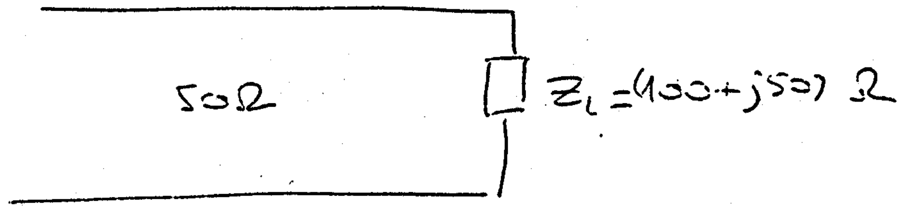
$S(1,\infty)$ $\rightarrow$ measure of the mismatch of load/line (real)

(·) Matched load: $\Gamma = 0 \Rightarrow S = 1$
$|\Gamma| = 1 \Rightarrow S = \infty$

Always $S \geq 1$

Power: $P_{av}^i = |V_0^+|^2 / 2Z_0$, $P_{av}^r = +|\Gamma|^2 |V_0^+|^2 / 2Z_0 = (V_0^-)^2 / 2Z_0 = |\Gamma|^2 P_{av}^i$

Example



$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50} =$$

$$= \frac{70.7 \angle 45^\circ}{158.1 \angle 18.4^\circ} = 0.45 \angle 26.6^\circ$$

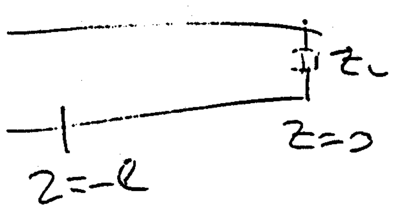
$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.45}{1 - 0.45} = 2.6$$

Input Impedance of the Lossless Line

$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \left[\frac{1 + \Gamma e^{j2\beta z}}{1 - \Gamma e^{j2\beta z}} \right]$$

→ ratio of total voltage (incident + reflected) to the total current at any point z of the trans. line with char. impedance Z_0 and load Z_L.

Convention:



$$Z_{in}(-l) = Z_0 \left(\frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} \right) = Z_0 \left(\frac{Z_L \cos\beta l + j Z_0 \sin\beta l}{Z_0 \cos\beta l + j Z_L \sin\beta l} \right) =$$

$$\star = Z_0 \left(\frac{Z_L + j Z_0 \tan\beta l}{Z_0 + j Z_L \tan\beta l} \right)$$

Example of (3)

$SC \rightarrow Z_L = 0$

$Z_{in}(z=-\ell) = jZ_0 \tan\beta\ell$

+ - inductor - - capacitor

$f = 1.05\,GHz$ — $Z_g = 10\,\Omega$

$Z_L = (100 + j50)\,\Omega$, $Z_0 = 50\,\Omega$ (default value)

connected through a 67cm-long lossless TRL. Teflon $\varepsilon_r = 2.04$

Phase velocity on the line: $0.7\,c_0 = 0.7 \times 3 \times 10^8$ m/sec

Is it matched to the generator?

Need to find: $Z_{in}(z = -0.67m)$

$$\lambda = \frac{u_p}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2m \qquad V_p = c/\sqrt{\varepsilon_r}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.2}$$

$\ell = \frac{\lambda}{4}$ (electrical length)

$\ell = k\lambda \qquad \ell = \frac{.67}{.2} = 3.35\lambda$

$$\beta\ell = \frac{2\pi}{\lambda}\ell = \frac{2\pi}{0.2}0.67 \text{ rads} \Rightarrow \tan(\beta\ell) = \tan(6.7\pi)$$

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0\tan\beta\ell}{Z_0 + jZ_L\tan\beta\ell}\right] = 50\left[\frac{100 + j50 + j50\tan(6.7\pi)}{50 + (100 + j50)\tan(6.7\pi)}\right]$$

$$= (21.9 + j17.4)\,\Omega \neq Z_g \;!!$$

$P_{refl} = |\Gamma_0|^2 P_{in}$

Short-circuit load

$\Gamma = -1 \Rightarrow Z_{in}^{sc} = jZ_0\tan\beta\ell \rightarrow$ Pure Imaginary

If $\tan\beta\ell \geq 0 \Rightarrow$ equiv. inductance: $j\omega L_{eq} = jZ_0\tan\beta\ell$

If $\tan\beta\ell \leq 0 \Rightarrow$ equiv. capacitance: $\frac{1}{j\omega C_{eq}} = jZ_0\tan\beta\ell$

// Open-circuited load: $Z_{in}^{oc} = -jZ_0\cot\beta\ell$

A network analyzer is an RF instrument that measures the impedance of any load connected to its input terminal.

$$Z_0 = \sqrt{Z_{in}^{sc}\, Z_{in}^{oc}}$$ (Determination of Z_0 through Z_{in}^{sc}, Z_{in}^{oc})

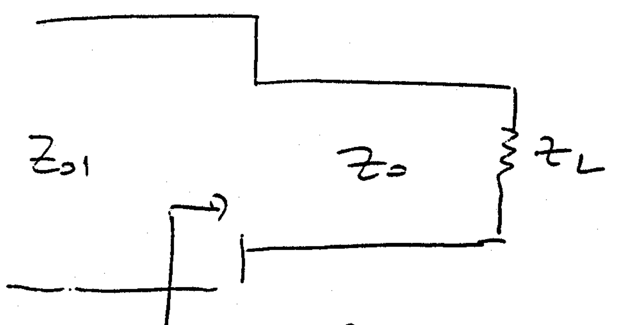
Lines of Length $l = n\lambda/2$

$\tan \beta l = \tan n\pi = 0 \Rightarrow Z_{in} = Z_L$

$\frac{2\pi}{\lambda} l \left(\frac{\lambda}{2}\right)$ $\beta l = \frac{\pi}{2} + n\pi = \frac{2\pi l}{\lambda}$

Quarter-Wave Transformer $\boxed{Z_{in}\left(z = -n\frac{\lambda}{2}\right) = Z_L}$

$l = \frac{\lambda}{4} + n\frac{\lambda}{2}$ $\quad Z_{in} = \cancel{Z_0}\frac{Z_L}{\cancel{Z_0}}$



$Z_{in} = \frac{Z_0^2}{Z_L} = Z_{01}$ (Matching of Z_L to Z_{01})

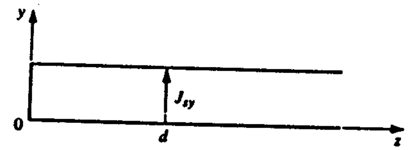
$Z_0 = \sqrt{Z_{01} Z_L}$ $\quad Z_{01} = Z_{in}$

eg. $Z_L = 50\Omega \longrightarrow Z_{01} = 100\Omega$

$\lambda/4$ transformer with Z_0: $\quad 100 = \frac{Z_0^2}{50} \Rightarrow Z_0 = 70.7\Omega$

lossless line → Z_0 real, match only real loads w/ $\lambda/4$ works for only 1 freq. xfr.

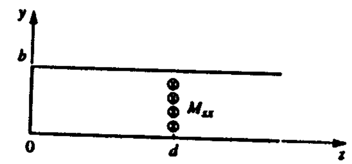
and removing the shorting wall at $z = 0$. Use the results of Section 5.9 and superposition to find the fields radiated by these two currents, which should be the same as the first results for $z > 0$.

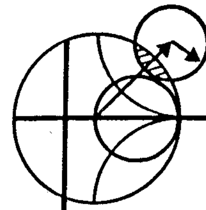


5.43 A rectangular waveguide is shorted at $z = 0$, and has a magnetic current sheet, M_{sx}, located at $z = d$, where

$$M_{sx} = \frac{2\pi B}{a} \sin\frac{\pi x}{a}.$$

Find the fields radiated by this current for $z > 0$. What is the correct image current that should be placed at $z = -d$, to account for the presence of the conducting wall at $z = 0$?





CHAPTER 6

Impedance Matching and Tuning

This chapter marks a turning point in that we now begin to apply the theory and techniques of the previous chapters to practical problems in microwave engineering. We begin with the topic of impedance matching, which is often a part of the larger design process for a microwave component or system. The basic idea of impedance matching is illustrated in Figure 6.1, which shows an impedance matching network placed between a load impedance and a transmission line. The matching network is ideally lossless, to avoid unnecessary loss of power, and is usually designed so that the impedance seen looking into the matching network is Z_0. Then reflections are eliminated on the transmission line to the left of the matching network, although there will be multiple reflections between the matching network and the load. This procedure is also referred to as tuning. Impedance matching or tuning is important for the following reasons: tuning

(+) tuning

- Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in the feed line is minimized.
- Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc.) improves the signal-to-noise ratio of the system.
- Impedance matching in a power distribution network (such as an antenna array feed network) will reduce amplitude and phase errors.

As long as the load impedance, Z_L, has some nonzero real part, a matching network can always be found. Many choices are available, however, and we will discuss the design and performance of several types of practical matching networks. Factors that may be important in the selection of a particular matching network include the following:

- *Complexity*—As with most engineering solutions, the simplest design that satisfies the required specifications is generally the most preferable. A simpler matching

FIGURE 6.1 A lossless network matching an arbitrary load impedance to a transmission line.

network is usually cheaper, more reliable, and less lossy than a more complex design.

- *Bandwidth*—Any type of matching network can ideally give a perfect match (zero reflection) at a single frequency. In many applications, however, it is desirable to match a load over a band of frequencies. There are several ways of doing this with, of course, a corresponding increase in complexity.
- *Implementation*—Depending on the type of transmission line or waveguide being used, one type of matching network may be preferable compared to another. For example, tuning stubs are much easier to implement in waveguide than are multisection quarter-wave transformers.
- *Adjustability*—In some applications the matching network may require adjustment to match a variable load impedance. Some types of matching networks are more amenable than others in this regard.

6.1 MATCHING WITH LUMPED ELEMENTS (*L NETWORKS*)

section

Probably the simplest type of matching network is the L section, which uses two reactive elements to match an arbitrary load impedance to a transmission line. There are two possible configurations for this network, as shown in Figure 6.2. If the normalized load impedance, $z_L = Z_L/Z_0$, is inside the $1 + jx$ circle on the Smith chart, then the circuit of Figure 6.2a should be used. If the normalized load impedance is outside the $1 + jx$ circle on the Smith chart, the circuit of Figure 6.2b should be used. The $1 + jx$ circle is the resistance circle on the impedance Smith chart for which $r = 1$.

In either of the configurations of Figure 6.2, the reactive elements may be either inductors or capacitors, depending on the load impedance. Thus, there are eight distinct possibilities for the matching circuit for various load impedances. ~~If the frequency is low enough and/or the circuit size is small enough, actual lumped-element capacitors and inductors can be used.~~ This may be feasible for frequencies up to about 1 GHz or so, although modern microwave integrated circuits may be small enough so that lumped elements can be used at higher frequencies as well. There is, however, a large range of frequencies and circuit sizes where lumped elements may not be realizable. This is a limitation of the L section matching technique.

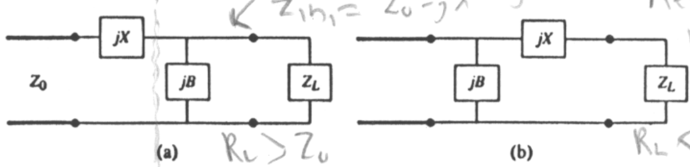
(-) L-section matching technique

Z_{in_1} $Z_0 = Z_{in_1} + jX$

$Z_{in_1} = Z_0 - jX = jB \| Z_L$ $\quad Re(Z_{in_1}) = Z_0$

$Re(jB \| Z_L) = Z_0$ $\quad B = ?$

Then solve for $X = ?$



$R_L > Z_0$ $\quad R_L < Z_0$

FIGURE 6.2 L section matching networks. (a) Network for z_L inside the $1 + jx$ circle. (b) Network for z_L outside the $1 + jx$ circle.

$R_L > Z_0$

$R_L < Z_0$

Can use O.C. or S.C. for lumped elements

We will now derive the analytic expressions for the matching network elements of the two cases in Figure 6.2, then illustrate an alternative design procedure using the Smith chart.

Analytic Solutions

Although we will discuss a simple graphical solution using the Smith chart, it may be useful to derive expressions for the L section matching network components. Such expressions would be useful in a computer-aided design program for L section matching, or when it is necessary to have more accuracy than the Smith chart can provide.

Consider first the circuit of Figure 6.2a, and let $Z_L = R_L + jX_L$. We stated that this circuit would be used when $z_L = Z_L/Z_0$ is inside the $1 + jx$ circle on the Smith chart, which implies that $R_L > Z_0$ for this case.

The impedance seen looking into the matching network followed by the load impedance must be equal to Z_0, for a match:

$$Z_0 = jX + \frac{1}{jB + (1/R_L + jX_L)}. \tag{6.1}$$

Rearranging and separating into real and imaginary parts gives two equations for the two unknowns, X and B:

$$B(XR_L - X_L Z_0) = R_L - Z_0, \tag{6.2a}$$

$$X(1 - BX_L) = BZ_0R_L - X_L. \tag{6.2b}$$

Solving (6.2a) for X and substituting into (6.2b) gives a quadratic equation for B. The solution is

$$\boxed{B = \frac{X_L \pm \sqrt{R_L/Z_0}\sqrt{R_L^2 + X_L^2 - Z_0R_L}}{R_L^2 + X_L^2}.} \tag{6.3a}$$

inside $1+jx$

Note that since $R_L > Z_0$, the argument of the second square root is always positive. Then the series reactance can be found as

$$X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L}. \tag{6.3b}$$

Equation (6.3a) indicates that two solutions are possible for B and X. Both of these solutions are physically realizable, since both positive and negative values of B and X are possible (positive X implies an inductor, negative X implies a capacitor, while positive B implies a capacitor and negative B implies an inductor.) One solution, however, may result in significantly smaller values for the reactive components, and may be the preferred solution if the bandwidth of the match is better, or the SWR on the line between the matching network and the load is smaller.

Now consider the circuit of Figure 6.2b. This circuit is to be used when z_L is outside the $1 + jx$ circle on the Smith chart, which implies that $R_L < Z_0$. The admittance seen looking into the matching network followed by the load impedance $Z_L = R_L + jX_L$

must be equal to $1/Z_0$, for a match:

$$\frac{1}{Z_0} = jB + \frac{1}{R_L + j(X + X_L)}. \tag{6.4}$$

Rearranging and separating into real and imaginary parts gives two equations for the two unknowns, X and B:

$$BZ_0(X + X_L) = Z_0 - R_L, \tag{6.5a}$$

$$(X + X_L) = BZ_0R_L. \tag{6.5b}$$

Solving for X and B gives

out side
1+jx

$$X = \pm\sqrt{R_L(Z_0 - R_L)} - X_L, \tag{6.6a}$$

$$B = \pm\frac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0}. \tag{6.6b}$$

Since $R_L < Z_0$, the arguments of the square roots are always positive. Again, note that two solutions are possible.

In order to match an arbitrary complex load to a line of characteristic impedance Z_0, the real part of the input impedance to the matching network must be Z_0, while the imaginary part must be zero. This implies that a general matching network must have at least two degrees of freedom; in the L section matching circuit these two degrees of freedom are provided by the values of the two reactive components.

Smith Chart Solutions

Instead of the above formulas, the Smith chart can be used to quickly and accurately design L section matching networks, a procedure best illustrated by an example.

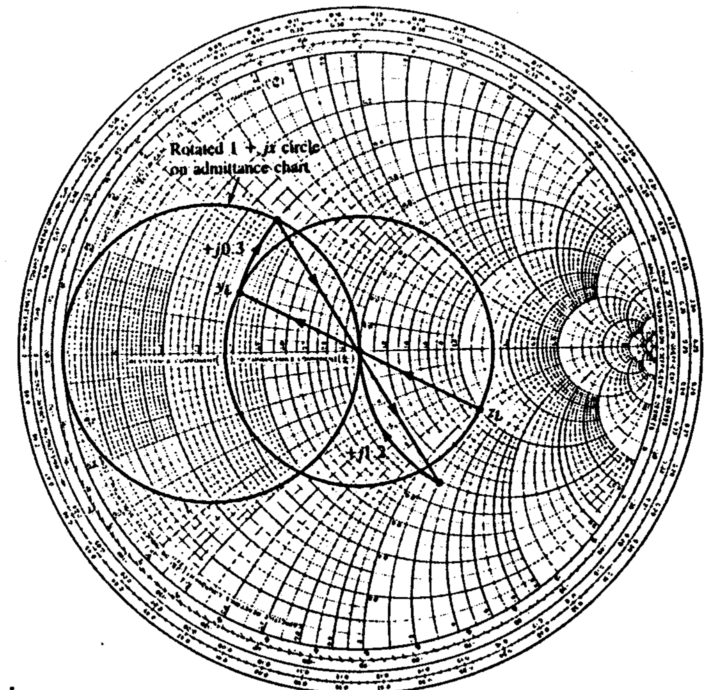
EXAMPLE 6.1

Design an L section matching network to match a series RC load with an impedance $Z_L = 200 - j100\ \Omega$, to a 100 Ω line, at a frequency of 500 MHz.

Solution

The normalized load impedance is $z_L = 2 - j1$, which is plotted on the Smith chart of Figure 6.3a. This point is inside the $1 + jx$ circle, so we will use the matching circuit of Figure 6.2a. Since the first element from the load is a shunt susceptance, it makes sense to convert to admittance by drawing the SWR circle through the load, and a straight line from the load through the center of the chart, as shown in Figure 6.3a. Now, after we add the shunt susceptance and convert back to impedance, we want to be on the $1 + jx$ circle, so that we can add a series reactance to cancel the jx and match the load. This means that the shunt susceptance must move us from y_L to the $1 + jx$ circle on the *admittance* Smith chart. Thus, we construct the rotated $1 + jx$ circle as shown in Figure 6.3a (center at 0.333). (A combined ZY chart is convenient to use here, if it is not too confusing.) Then we see that adding a susceptance of $jb = j0.3$ will move us along a constant conductance circle to $y = 0.4 + j0.5$ (this choice is

1
2



(a) *(continues)*

FIGURE 6.3 Solution to Example 6.1. (a) Smith chart for the L section matching networks.

the shortest distance from y_L to the shifted $1 + jx$ circle). Converting back to impedance leaves us at $z = 1 - j1.2$, indicating that a series reactance $x = j1.2$ will bring us to the center of the chart. For comparison, the formulas of (6.3a,b) give the solution as $b = 0.29, x = 1.22$.

This matching circuit consists of a shunt capacitor and a series inductor, as shown in Figure 6.3b. For a frequency of $f = 500$ MHz, the capacitor has a value of

$$C = \frac{b}{2\pi f Z_0} = 0.92\ \text{pF},$$

and the inductor has a value of

$$L = \frac{xZ_0}{2\pi f} = 38.8\ \text{nH}.$$

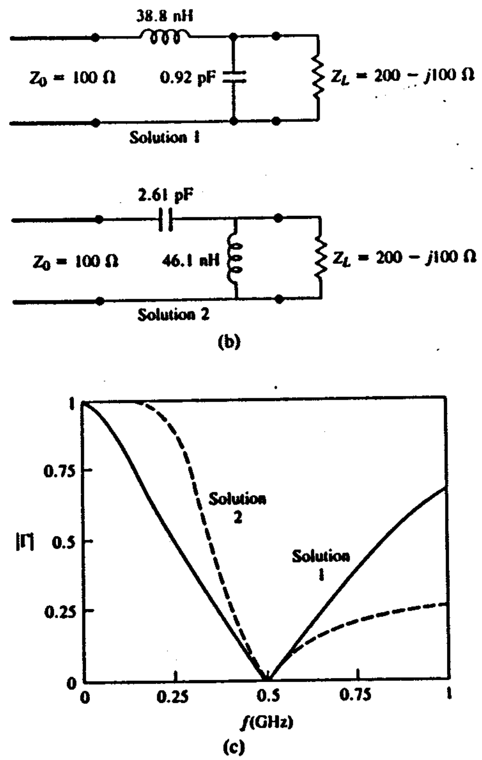


FIGURE 6.3 Continued. (b) The two possible L section matching circuits. (c) Reflection coefficient magnitudes versus frequency for the matching circuits of (b).

It may also be interesting to look at the second solution to this matching problem. If instead of adding a shunt susceptance of $b = 0.3$, we use a shunt susceptance of $b = -0.7$, we will move to a point on the lower half of the shifted $1 + jx$ circle, to $y = 0.4 - j0.5$. Then converting to impedance and adding a series reactance of $x = -1.2$ leads to a match as well. The formulas of (6.3a,b) give this solution as $b = -0.69, x = -1.22$. This matching circuit is also shown in Figure 6.3b, and is seen to have the positions of the inductor and capacitor reversed from the first matching network. At a frequency of $f = 500$ MHz, the capacitor has a value of

$$C = \frac{-1}{2\pi f x Z_0} = 2.61 \text{ pF},$$

while the inductor has a value of

$$L = \frac{-Z_0}{2\pi f b} = 46.1 \text{ nH}.$$

Figure 6.3c shows the reflection coefficient magnitude versus frequency for these two matching networks, assuming that the load impedance of $Z_L = 200 - j100\ \Omega$ at 500 MHz consists of a 200 Ω resistor and a 3.18 pF capacitor in series. There is not a substantial difference in bandwidth for these two solutions. ○

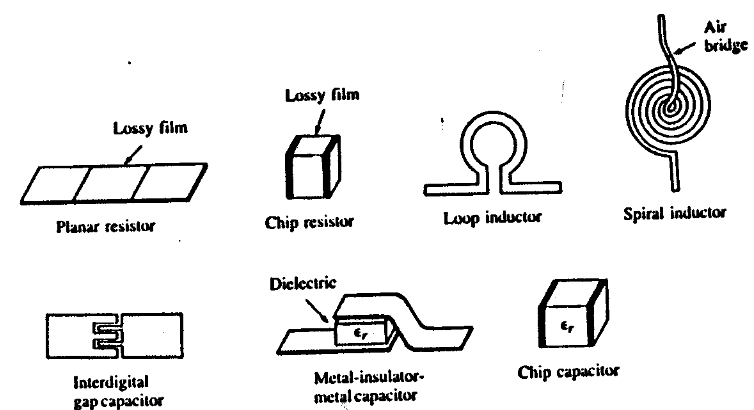
POINT OF INTEREST: Lumped Elements for Microwave Integrated Circuits

Lumped R, L, and C elements can be practically realized at microwave frequencies if the length, ℓ, of the component is very small relative to the operating wavelength. Over a limited range of values, such components can be used in hybrid and monolithic microwave integrated circuits (MICs) at frequencies up to 60 GHz, if the condition that $\ell < \lambda/10$ is satisfied. Usually, however, the characteristics of such an element are far from ideal, requiring that undesirable effects such as parasitic capacitance and/or inductance, spurious resonances, fringing fields, loss, and perturbations caused by a ground plane be incorporated in the design via a CAD model (see the Point of Interest concerning CAD).

Resistors are fabricated with thin films of lossy material such as nichrome, tantalum nitride, or doped semiconductor material. In monolithic circuits such films can be deposited or grown, while chip resistors made from a lossy film deposited on a ceramic chip can be bonded or soldered in a hybrid circuit. Low resistances are hard to obtain.

Small values of inductance can be realized with a short length or loop of transmission line, and larger values (up to about 10 nH) can be obtained with a spiral inductor, as shown in the following figures. Larger inductance values generally incur more loss, and more shunt capacitance; this leads to a resonance that limits the maximum operating frequency.

Capacitors can be fabricated in several ways. A short transmission line stub can provide a shunt capacitance in the range of 0 to 0.1 pF. A single gap or interdigital set of gaps in a transmission



variable C
planar

line can provide a series capacitance up to about 0.5 pF. Greater values (up to about 25 pF) can be obtained using a metal-insulator-metal (MIM) sandwich, either in monolithic or chip (hybrid) form.

6.2 SINGLE-STUB TUNING

We next consider a matching technique that uses a single open-circuited or short-circuited length of transmission line (a "stub"), connected either in parallel or in series with the transmission feed line at a certain distance from the load, as shown in Figure 6.4. Such a tuning circuit is convenient from a microwave fabrication aspect, since lumped elements are not required. The shunt tuning stub is especially easy to fabricate in microstrip or stripline form.

In single-stub tuning, the two adjustable parameters are the distance, d, from the load to the stub position, and the value of susceptance or reactance provided by the shunt or series stub. For the shunt-stub case, the basic idea is to select d so that the admittance,

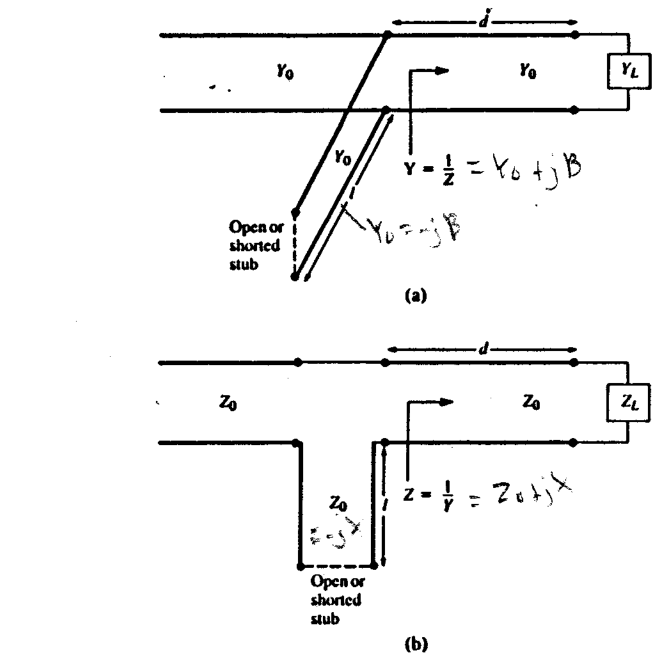


FIGURE 6.4 Single-stub tuning circuits. (a) Shunt stub. (b) Series stub.

$Z_{in}^{sc} = jZ_0 \tan\beta\ell$

$Z_{in}^{oc} = -jZ_0 \cot\beta\ell$

Y, seen looking into the line at distance d from the load is of the form $Y_0 + jB$. Then the stub susceptance is chosen as $-jB$, resulting in a matched condition. For the series stub case, the distance d is selected so that the impedance, Z, seen looking into the line at a distance d from the load is of the form $Z_0 + jX$. Then the stub reactance is chosen as $-jX$, resulting in a matched condition.

As discussed in Chapter 3, the proper length of open or shorted transmission line can provide any desired value of reactance or susceptance. For a given susceptance or reactance, the difference in lengths of an open- or short-circuited stub is $\lambda/4$. For transmission line media such as microstrip or stripline, open-circuited stubs are easier to fabricate since a via hole through the substrate to the ground plane is not needed. For lines like coax or waveguide, however, short-circuited stubs are usually preferred, because the cross-sectional area of such an open-circuited line may be large enough (electrically) to radiate, in which case the stub is no longer purely reactive.

Below we discuss both Smith chart and analytic solutions for shunt and series stub tuning. The Smith chart solutions are fast, intuitive, and usually accurate enough in practice. The analytic expressions are more accurate, and useful for computer analysis.

Shunt Stubs

The single-stub shunt tuning circuit is shown in Figure 6.4a. We will first discuss an example illustrating the Smith chart solution, and then derive formulas for d and ℓ.

EXAMPLE 6.2

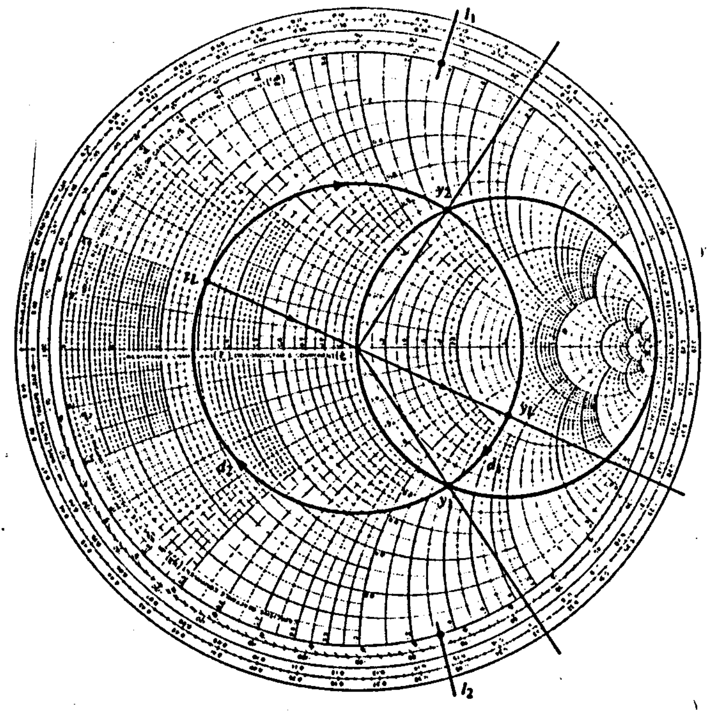
For a load impedance $Z_L = 15 + j10\ \Omega$, design two single-stub shunt tuning networks to match this load to a 50 Ω line. Assuming that the load is matched at 2 GHz, and that the load consists of a resistor and inductor in series, plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

Solution

The first step is to plot the normalized load impedance $z_L = 0.3 + j0.2$, construct the appropriate SWR circle, and convert to the load admittance, y_L, as shown on the Smith chart in Figure 6.5a. For the remaining steps we consider the Smith chart as an admittance chart. Now notice that the SWR circle intersects the $1 + jb$ circle at two points, denoted as y_1 and y_2 in Figure 6.5a. Thus the distance d, from the load to the stub, is given by either of these two intersections. Reading the WTG scale, we obtain

$$d_1 = 0.328 - 0.284 = 0.044\lambda,$$
$$d_2 = (0.5 - 0.284) + 0.171 = 0.387\lambda.$$

Actually, there are an infinite number of distances, d, on the SWR circle that intersect the $1 + jb$ circle. Usually, it is desired to keep the matching stub as close as possible to the load, to improve the bandwidth of the match and to reduce losses caused by a possibly large standing wave ratio on the line between the stub and the load.



(a) *(continues)*

FIGURE 6.5 Solution to Example 6.2. (a) Smith chart for the shunt-stub tuners.

At the two intersection points, the normalized admittances are

$$y_1 = 1 - j1.33,$$
$$y_2 = 1 + j1.33.$$

Thus, the first tuning solution requires a stub with a susceptance of $j1.33$. The length of an open-circuited stub that gives this susceptance can be found on the Smith chart by starting at $y = 0$ (the open circuit) and moving along the outer edge of the chart ($g = 0$) towards the generator to the $j1.33$ point. The length is then

$$\ell_1 = 0.147\lambda.$$

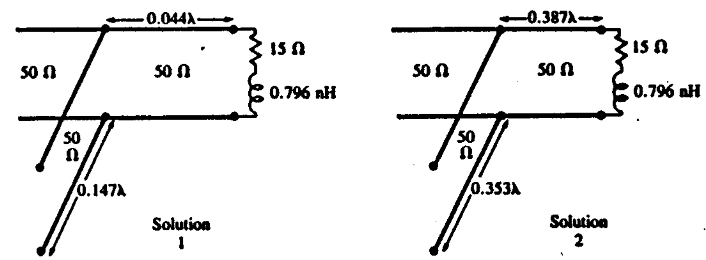
Similarly, the required open-circuit stub length for the second solution is

$$\ell_2 = 0.353\lambda.$$

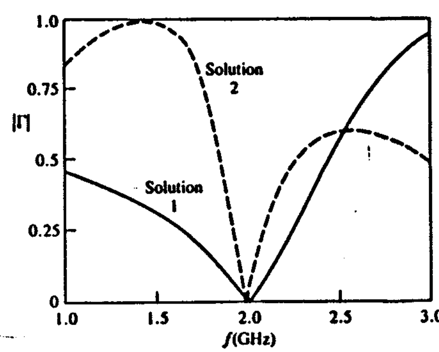
This completes the tuner designs.

To analyze the frequency dependence of these two designs, we need to know the load impedance as a function of frequency. The series-RL load impedance is $Z_L = 15 + j10\ \Omega$ at 2 GHz, so $R = 15\ \Omega$ and $L = 0.796$ nH. The two tuning circuits are shown in Figure 6.5b. Figure 6.5c shows the calculated reflection coefficient magnitudes for these two solutions. Observe that solution 1 has a significantly better bandwidth than solution 2; this is because both d and ℓ are shorter for solution 1, which reduces the frequency variation of the match. ○

Shorter d, ℓ ⇓ better BW



(b)



(c)

FIGURE 6.5 Continued. (b) The two shunt-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

To derive formulas for d and ℓ, let the load impedance be written as $Z_L = 1/Y_L = R_L + jX_L$. Then the impedance Z down a length, d, of line from the load is

$$Z = Z_0 \frac{(R_L + jX_L) + jZ_0 t}{Z_0 + j(R_L + jX_L)t}, \qquad 6.7$$

where $t = \tan\beta d$. The admittance at this point is

$$Y = G + jB = \frac{1}{Z},$$

where

$$G = \frac{R_L(1+t^2)}{R_L^2 + (X_L + Z_0 t)^2}, \qquad 6.8a$$

$$B = \frac{R_L^2 t - (Z_0 - X_L t)(X_L + Z_0 t)}{Z_0[R_L^2 + (X_L + Z_0 t)^2]}. \qquad 6.8b$$

Now d (which implies t) is chosen so that $G = Y_0 = 1/Z_0$. From (6.8a), this results in a quadratic equation for t:

$$Z_0(R_L - Z_0)t^2 - 2X_L Z_0 t + (R_L Z_0 - R_L^2 - X_L^2) = 0.$$

Solving for t gives

$$\boxed{t = \frac{X_L \pm \sqrt{R_L[(Z_0 - R_L)^2 + X_L^2]/Z_0}}{R_L - Z_0}, \quad \text{for } R_L \neq Z_0} = \tan(\beta d) \qquad 6.9$$

If $R_L = Z_0$, then $t = -X_L/2Z_0$. Thus, the two principal solutions for d are

$$\frac{d}{\lambda} = \begin{cases} \dfrac{1}{2\pi}\tan^{-1} t & \text{for } t \geq 0 \\ \dfrac{1}{2\pi}(\pi + \tan^{-1} t) & \text{for } t < 0. \end{cases} \Big\} \text{ for } R_L = Z_0 \qquad 6.10$$

To find the required stub lengths, first use t in (6.8b) to find the stub susceptance, $B_s = -B$. Then, for an open-circuited stub,

$$\frac{\ell_o}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{B_s}{Y_0}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{B}{Y_0}\right), \Big\} \text{ o.c. stub} \qquad 6.11a$$

while for a short-circuited stub,

$$\frac{\ell_s}{\lambda} = \frac{-1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B_s}\right) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B}\right). \Big\} \text{ s.c. stub} \qquad 6.11b$$

If the length given by (6.11a) or (6.11b) is negative, $\lambda/2$ can be added to give a positive result.

Series Stubs

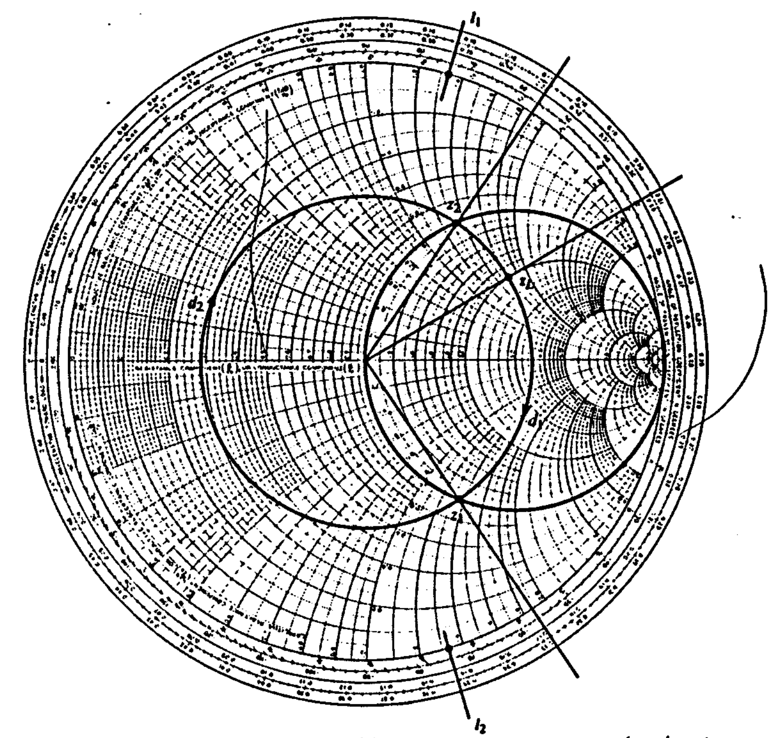
The series stub tuning circuit is shown in Figure 6.4b. We will illustrate the Smith chart solution by an example, and then derive expressions for d and ℓ.

EXAMPLE 6.3

Match a load impedance of $Z_L = 100 + j80$ to a 50 Ω line using a single series open-circuit stub. Assuming that the load is matched at 2 GHz, and that the load consists of a resistor and inductor in series, plot the reflection coefficient magnitude from 1 GHz to 3 GHz.

Solution

The first step is to plot the normalized load impedance, $z_L = 2 + j1.6$, and draw the SWR circle. For the series-stub design, the chart is an impedance chart. Note that the SWR circle intersects the $1 + jx$ circle at two points, denoted as z_1 and z_2 in Figure 6.6a. The shortest distance, d_1, from the load to



(a) *(continues)*

FIGURE 6.6 Solution to Example 6.3. (a) Smith chart for the series-stub tuners.

the stub is, from the WTG scale,

$$d_1 = 0.328 - 0.208 = 0.120\lambda,$$

while the second distance is

$$d_2 = (0.5 - 0.208) + 0.172 = 0.463\lambda.$$

As in the shunt-stub case, additional rotations around the SWR circle lead to additional solutions, but these are usually not of practical interest.

The normalized impedances at the two intersection points are

$$z_1 = 1 - j1.33,$$
$$z_2 = 1 + j1.33.$$

Thus, the first solution requires a stub with a reactance of $j1.33$. The length of an open-circuited stub that gives this reactance can be found on the Smith chart by starting at $z = \infty$ (open circuit), and moving along the outer edge of the chart ($r = 0$) toward the generator to the $j1.33$ point. This gives a stub length of

$$\ell_1 = 0.397\lambda.$$

Similarly, the required open-circuited stub length for the second solution is

$$\ell_2 = 0.103\lambda.$$

This completes the tuner designs.

If the load is a series resistor and inductor with $Z_L = 100 + j80\ \Omega$ at 2 GHz, then $R = 100\ \Omega$ and $L = 6.37$ nH. The two matching circuits are shown in Figure 6.6b. Figure 6.6c shows the calculated reflection coefficient magnitudes versus frequency for the two solutions. ○

To derive formulas for d and ℓ for the series-stub tuner, let the load admittance be written as $Y_L = 1/Z_L = G_L + jB_L$. Then the admittance Y down a length, d, of line from the load is

$$Y = Y_0 \frac{(G_L + jB_L) + jtY_0}{Y_0 + jt(G_L + jB_L)}, \qquad 6.12$$

where $t = \tan\beta d$, and $Y_0 = 1/Z_0$. Then the impedance at this point is

$$Z = R + jX = \frac{1}{Y},$$

where

$$R = \frac{G_L(1 + t^2)}{G_L^2 + (B_L + Y_0 t)^2}, \qquad 6.13a$$

$$X = \frac{G_L^2 t - (Y_0 - tB_L)(B_L + tY_0)}{Y_0[G_L^2 + (B_L + Y_0 t)^2]}. \qquad 6.13b$$

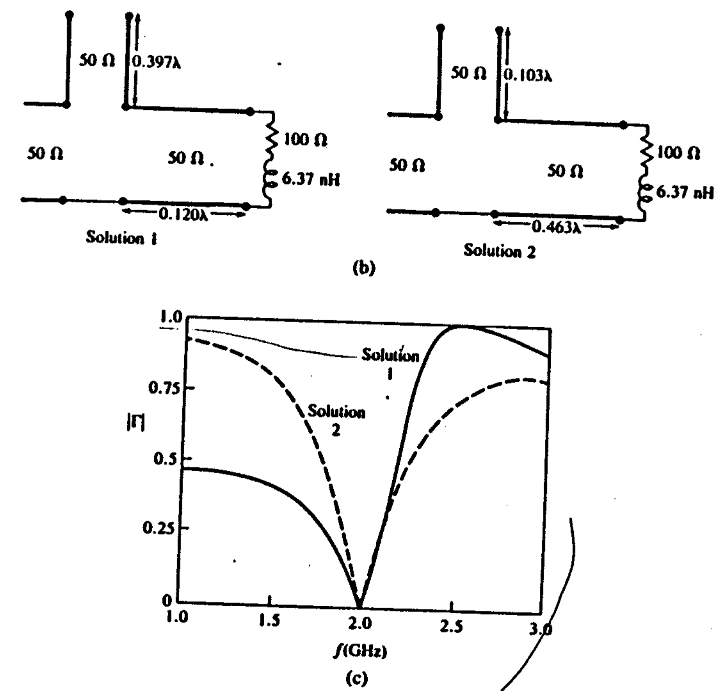


FIGURE 6.6 Continued. (b) The two series-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning curcuits of (b).

Now d (which implies t) is chosen so that $R = Z_0 = 1/Y_0$. From (6.13a), this results in a quadratic equation for t:

$$Y_0(G_L - Y_0)t^2 - 2B_L Y_0 t + (G_L Y_0 - G_L^2 - B_L^2) = 0.$$

Solving for t gives

$$\boxed{t = \frac{B_L \pm \sqrt{G_L[(Y_0 - G_L)^2 + B_L^2]/Y_0}}{G_L - Y_0}, \quad \text{for } G_L \neq Y_0.} \qquad 6.14$$

If $G_L = Y_0$, then $t = -B_L/2Y_0$. Then the two principal solutions for d are

$$d/\lambda = \begin{cases} \dfrac{1}{2\pi}\tan^{-1} t & \text{for } t \geq 0 \\ \dfrac{1}{2\pi}(\pi + \tan^{-1} t) & \text{for } t < 0. \end{cases} \qquad 6.15$$

The required stub lengths are determined by first using t in (6.13b) to find the reactance, X. This reactance is the negative of the necessary stub reactance, X_s. Thus, for a short-circuited stub,

$$\frac{\ell_s}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{X_s}{Z_0}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{X}{Z_0}\right), \qquad 6.16a$$

while for an open-circuited stub,

$$\frac{\ell_o}{\lambda} = \frac{-1}{2\pi}\tan^{-1}\left(\frac{Z_0}{X_s}\right) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Z_0}{X}\right). \qquad 6.16b$$

If the length given by (6.16a) or (6.16b) is negative, $\lambda/2$ can be added to give a positive result.

6.3 DOUBLE-STUB TUNING

The single-stub tuners of the previous section are able to match any load impedance (as long as it has a nonzero real part) to a transmission line, but suffer from the disadvantage of requiring a variable length of line between the load and the stub. This may not be a problem for a fixed matching circuit, but would probably pose some difficulty if an adjustable tuner was desired. In this case, the double-stub tuner, which uses two tuning stubs in fixed positions, can be used. Such tuners are often fabricated in coaxial line, with adjustable stubs connected in parallel to the main coaxial line. We will see, however, that the double-stub tuner cannot match all load impedances.

The double-stub tuner circuit is shown in Figure 6.7a, where the load may be an arbitrary distance from the first stub. Although this is more representative of a practical situation, the circuit of Figure 6.7b, where the load Y_L' has been transformed back to the position of the first stub, is easier to deal with and does not lose any generality. The stubs shown in Figure 6.7 are shunt stubs, which are usually easier to implement in practice than are series stubs; the latter could be used just as well, in principle. In either case, the stubs can be open-circuited or short-circuited.

Smith Chart Solution

The Smith chart of Figure 6.8 illustrates the basic operation of the double-stub tuner. As in the case of the single-stub tuners, two solutions are possible. The susceptance of the first stub, b_1 (or b_1', for the second solution), moves the load admittance to y_1 (or y_1'). These points lie on the rotated $1 + jb$ circle; the amount of rotation is d wavelengths toward the load, where d is the electrical distance between the two stubs. Then transforming y_1 (or y_1') toward the generator through a length, d, of line leaves us at the point y_2 (or y_2'), which must be on the $1 + jb$ circle. The second stub then adds a susceptance b_2 (or b_2'), which brings us to the center of the chart, and completes the match.

Notice from Figure 6.8 that if the load admittance, y_L, were inside the shaded region of the $g_0 + jb$ circle, no value of stub susceptance b_1 could ever bring the load point to intersect the rotated $1 + jb$ circle. This shaded region thus forms a forbidden range of load admittances, which cannot be matched with this particular double-stub tuner. A simple way of reducing the forbidden range is to reduce the distance, d, between the stubs. This has the effect of swinging the rotated $1 + jb$ circle back towards the $y = \infty$ point, but d must be kept large enough for the practical purpose of fabricating the two separate stubs. In addition, stub spacings near 0 or $\lambda/2$ lead to matching networks that are very frequency sensitive. In practice, stub spacings are usually chosen as $\lambda/8$ or $3\lambda/8$. If the length of line between the load and the first stub can be adjusted, then the load admittance y_L can always be moved out of the forbidden region.

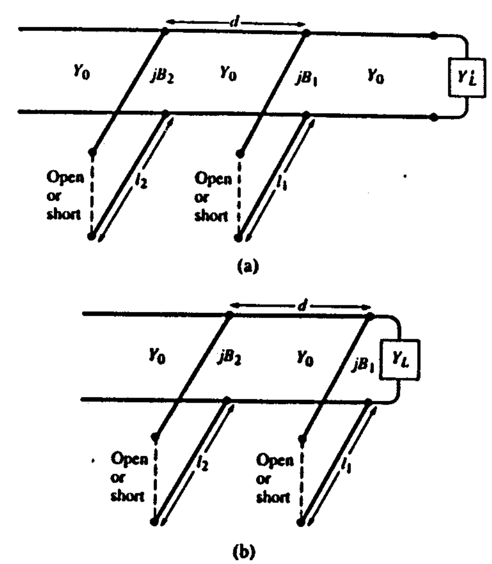


FIGURE 6.7 Double-stub tuning. (a) Original circuit with the load an arbitrary distance from the first stub. (b) Equivalent circuit with load at the first stub.

EXAMPLE 6.4

Design a double-stub shunt tuner to match a load impedance $Z_L = 60 - j80\ \Omega$ to a 50 Ω line. The stubs are to be short-circuited stubs, and are spaced $\lambda/8$ apart. Assuming that this load consists of a series resistor and capacitor, and that the match frequency is 2 GHz, plot the reflection coefficient magnitude versus frequency from 1 GHz to 3 GHz.

Solution

The normalized load admittance is $y_L = 0.3 + j0.4$, which is plotted on the Smith chart of Figure 6.9a. Next we construct the rotated $1 + jb$ conductance circle, by moving every point on the $g = 1$ circle $\lambda/8$ toward the load. We then

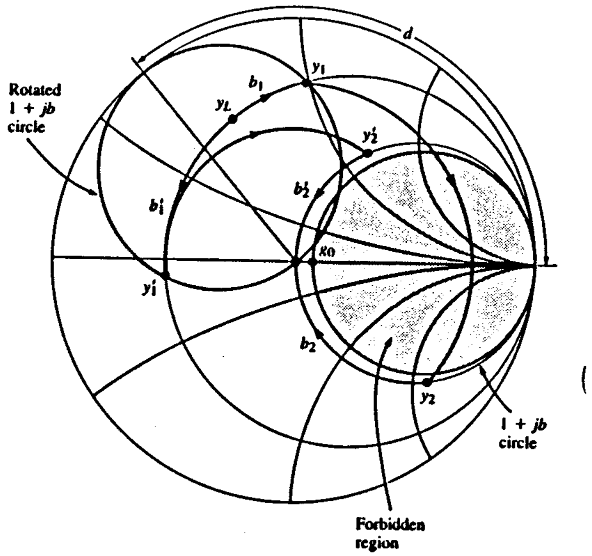


FIGURE 6.8 Smith chart diagram for the operation of a double-stub tuner.

find the susceptance of the first stub, which can be one of two possible values:

$$b_1 = 1.314,$$

or

$$b_1' = -0.114.$$

We now transform through the $\lambda/8$ section of line by rotating along a constant radius (SWR) circle $\lambda/8$ toward the generator. This brings the two solutions to the following points:

$$y_2 = 1 - j3.38,$$

or

$$y_2' = 1 + j1.38.$$

Then the susceptance of the second stub should be

$$b_2 = 3.38,$$

or

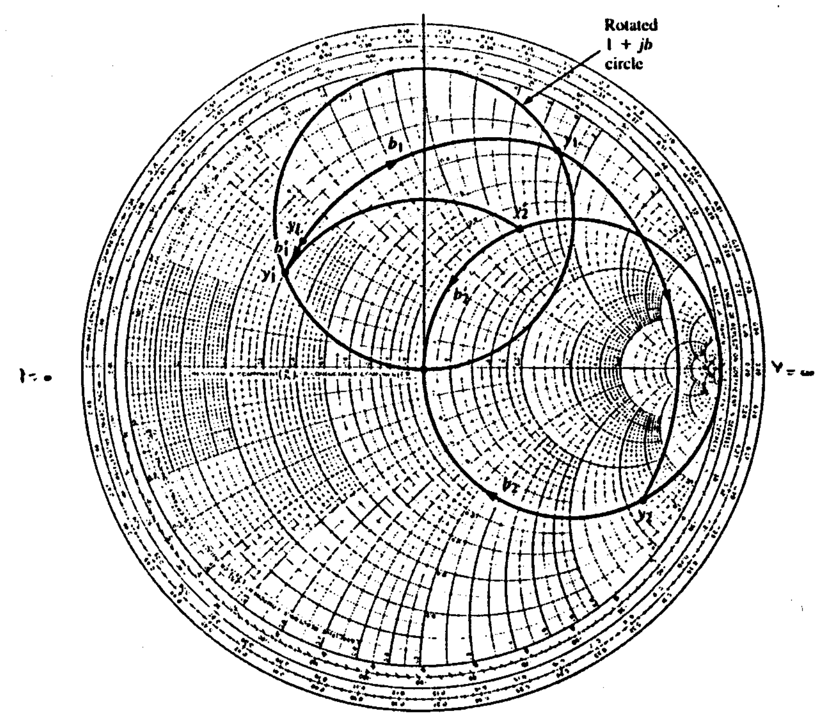
$$b_2' = -1.38.$$

The lengths of the short-circuited stubs are then found as

$$\ell_1 = 0.396\lambda, \quad \ell_2 = 0.454\lambda,$$

or

$$\ell_1' = 0.232\lambda, \quad \ell_2' = 0.100\lambda.$$



(a) *(continues)*

FIGURE 6.9 Solution to Example 6.4. (a) Smith chart for the double-stub tuners.

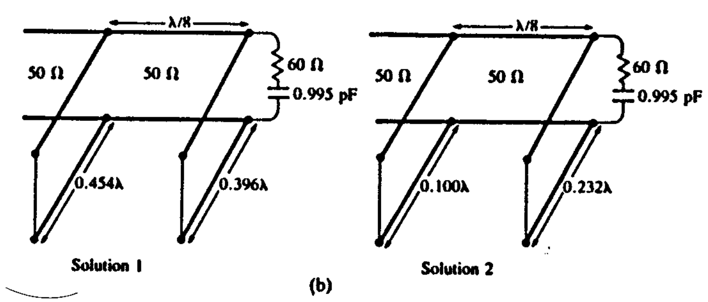
This completes both solutions for the double-stub tuner design.

Now if the resistor-capacitor load $Z_L = 60 - j80\ \Omega$ at $f = 2$ GHz, then $R = 60\ \Omega$ and $C = 0.995$ pF. The two tuning circuits are then shown in Figure 6.9b, and the reflection coefficient magnitudes are plotted versus frequency in Figure 6.9c. Note that the first solution has a much narrower bandwidth than the second (primed) solution, due to the fact that both stubs for the first solution are somewhat longer (and closer to $\lambda/2$) than the stubs of the second solution. ○

Analytic Solution

Just to the left of the first stub in Figure 6.7b, the admittance is

$$Y_1 = G_L + j(B_L + B_1), \tag{6.17}$$



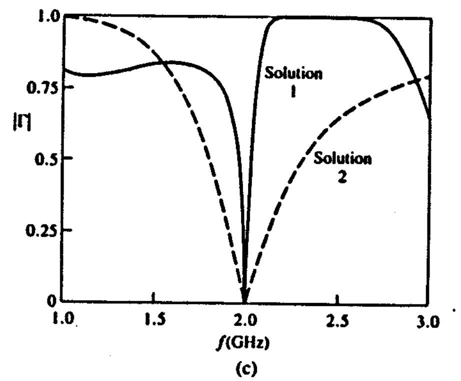


FIGURE 6.9 Continued. (b) The two double-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

where $Y_L = G_L + jB_L$ is the load admittance and B_1 is the susceptance of the first stub. After transforming through a length d of transmission line, the admittance just to the right of the second stub is

$$Y_2 = Y_0 \frac{G_L + j(B_L + B_1 + Y_0 t)}{Y_0 + jt(G_L + jB_L + jB_1)}, \tag{6.18}$$

where $t = \tan \beta d$ and $Y_0 = 1/Z_0$. At this point, the real part of Y_2 must equal Y_0, which leads to the equation

$$G_L^2 - G_L Y_0 \frac{1+t^2}{t^2} + \frac{(Y_0 - B_L t - B_1 t) Y_0}{t^2} = 0. \tag{6.19}$$

Solving for G_L gives

$$G_L = Y_0 \frac{1+t^2}{2t^2}\left[1 \pm \sqrt{1 - \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0(1+t^2)^2}}\right]. \tag{6.20}$$

Since G_L is real, the quantity within the square root must be nonnegative, and so

$$0 \le \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0(1+t^2)^2} \le 1.$$

This implies that

$$\boxed{0 \le G_L \le Y_0 \frac{1+t^2}{2t^2} = \frac{Y_0}{\sin^2 \beta d},} \tag{6.21}$$

which gives the range on G_L that can be matched for a given stub spacing, d. After d has been fixed, the first stub susceptance can be determined from (6.19) as

$$B_1 = -B_L + \frac{Y_0 \pm \sqrt{(1+t^2) G_L Y_0 - G_L^2 t^2}}{t}. \tag{6.22}$$

Then the second stub susceptance can be found from the negative of the imaginary part of (6.18) to be

$$B_2 = \frac{\pm Y_0 \sqrt{Y_0 G_L (1+t^2) - G_L^2 t^2} + G_L Y_0}{G_L t}. \tag{6.23}$$

The upper and lower signs in (6.22) and (6.23) correspond to the same solutions. The open-circuited stub length is found as

$$\frac{\ell_o}{\lambda} = \frac{1}{2\pi} \tan^{-1}\left(\frac{B}{Y_0}\right), \tag{6.24a}$$

o.c. stub

while the short-circuited stub length is found as

$$\frac{\ell_s}{\lambda} = \frac{-1}{2\pi} \tan^{-1}\left(\frac{Y_0}{B}\right), \tag{6.24b}$$

s.c. stub

where $B = B_1, B_2$.

6.4 THE QUARTER-WAVE TRANSFORMER

As discussed in Section 3.5, the quarter-wave transformer is a simple and useful circuit for matching a real load impedance to a transmission line. An additional feature of the quarter-wave transformer is that it can be extended to multisection designs in a methodical manner, for broader bandwidth. If only a narrow band impedance match is required, a single-section transformer may suffice. But, as we will see in the next few sections, multisection quarter-wave transformer designs can be synthesized to yield optimum matching characteristics over a desired frequency band. We will see in Chapter 9 that such networks are closely related to bandpass filters.

One drawback of the quarter-wave transformer is that it can only match a real load impedance. A complex load impedance can always be transformed to a real impedance, however, by using an appropriate length of transmission line between the load and the transformer, or an appropriate series or shunt reactive stub. These techniques will usually alter the frequency dependence of the equivalent load, which often has the effect of reducing the bandwidth of the match.

In Section 3.5 we analyzed the operation of the quarter-wave transformer from an impedance viewpoint and a multiple reflection viewpoint. Here we will concentrate on the bandwidth performance of the transformer, as a function of the load mismatch; this discussion will also serve as a prelude to the more general case of multisection transformers in the sections to follow.

The single-section quarter wave matching transformer circuit is shown in Figure 6.10. The characteristic impedance of the matching section is

$$\boxed{Z_1 = \sqrt{Z_0 Z_L}.} \qquad 6.25$$

At the design frequency, f_0, the electrical length of the matching section is $\lambda_0/4$, but at other frequencies the length is different, so a perfect match is no longer obtained. We will now derive an approximate expression for the mismatch versus frequency.

The input impedance seen looking into the matching section is

$$Z_{in} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}, \qquad 6.26$$

where $t = \tan \beta\ell = \tan \theta$, and $\beta\ell = \theta = \pi/2$ at the design frequency, f_0. The reflection coefficient is then

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_1(Z_L - Z_0) + jt(Z_1^2 - Z_0 Z_L)}{Z_1(Z_L + Z_0) + jt(Z_1^2 + Z_0 Z_L)}. \qquad 6.27$$

Since $Z_1^2 = Z_0 Z_L$, this reduces to

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}}. \qquad 6.28$$

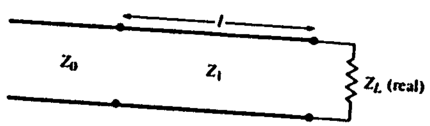


FIGURE 6.10 A single-section quarter-wave matching transformer. $\ell = \lambda_0/4$ at the design frequency f_0.

The reflection coefficient magnitude is

$$|\Gamma| = \frac{|Z_L - Z_0|}{\left[(Z_L + Z_0)^2 + 4t^2 Z_0 Z_L\right]^{1/2}}$$
$$= \frac{1}{\left\{(Z_L + Z_0/Z_L - Z_0)^2 + [4t^2 Z_0 Z_L/(Z_L - Z_0)^2]\right\}^{1/2}}$$
$$= \frac{1}{\left\{1 + [4Z_0 Z_L/(Z_L - Z_0)^2] + [4Z_0 Z_L t^2/(Z_L - Z_0)^2]\right\}^{1/2}}$$
$$= \frac{1}{\left\{1 + [4Z_0 Z_L/(Z_L - Z_0)^2] \sec^2\theta\right\}^{1/2}}, \qquad 6.29$$

since $1 + t^2 = 1 + \tan^2\theta = \sec^2\theta$.

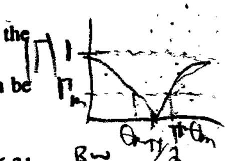
Now if we assume that the frequency is near the design frequency, f_0, then $\ell \simeq \lambda_0/4$ and $\theta \simeq \pi/2$. Then $\sec^2\theta \gg 1$, and (6.29) simplifies to

$f \approx f_0$

$$\boxed{|\Gamma| \simeq \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} |\cos\theta|, \quad \text{for } \theta \text{ near } \pi/2.} \qquad 6.30$$

Define Γ_m (=.01)

This result gives the approximate mismatch of the quarter-wave transformer near the design frequency, as sketched in Figure 6.11.



If we set a maximum value, Γ_m, of the reflection coefficient magnitude that can be tolerated, then we can define the bandwidth of the matching transformer as

$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right), \qquad 6.31$$

BW

$\Delta\theta = \beta\ell$

since the response of (6.29) is symmetric about $\theta = \pi/2$, and $\Gamma = \Gamma_m$ at $\theta = \theta_m$ and at $\theta = \pi - \theta_m$. Equating Γ_m to the exact expression for reflection coefficient magnitude in (6.29) allows us to solve for θ_m:

$$\frac{1}{\Gamma_m^2} = 1 + \left(\frac{2\sqrt{Z_0 Z_L}}{Z_L - Z_0} \sec\theta\right)^2,$$

or

$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}. \qquad 6.32$$

If we assume TEM lines, then

$$\theta = \beta\ell = \frac{2\pi f}{v_p} \frac{v_p}{4 f_0} = \frac{\pi f}{2 f_0},$$

TEM lines

therefore the frequency of the lower band edge at $\theta = \theta_m$ is

$$f_m = \frac{2\theta_m f_0}{\pi},$$

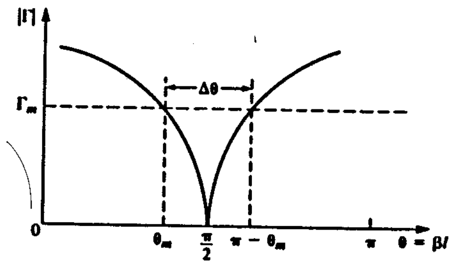


FIGURE 6.11 Approximate behavior of the reflection coefficient magnitude for a single-section quarter-wave transformer operating near its design frequency.

and the fractional bandwidth is, using (6.32), $\Delta f = f_2 - f_1$

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{2f_m}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= \boxed{2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right]}. \qquad 6.33$$

fractional BW

The fractional bandwidth is usually expressed as a percentage, $100\Delta f/f_0$ %. Note that the bandwidth of the transformer increases as Z_L becomes closer to Z_0 (a less mismatched load).

The above results are strictly valid only for TEM lines. When non-TEM lines (such as waveguides) are used, the propagation constant is no longer a linear function of frequency, and the wave impedance will be frequency dependent. These factors serve to complicate the general behavior of quarter-wave transformers for non-TEM lines, but in practice the bandwidth of the transformer is often small enough so that these complications do not substantially affect the result. Another factor ignored in the above analysis is the effect of reactances associated with discontinuities when there is a step change in the dimensions of a transmission line. This can often be compensated for by making a small adjustment in the length of the matching section.

non-TEM lines

Figure 6.12 shows a plot of the reflection coefficient magnitude versus normalized frequency for various mismatched loads. Note the trend of increased bandwidth for smaller load mismatches.

EXAMPLE 6.5

Design a single-section quarter-wave matching transformer to match a 10 Ω load to a 50 Ω line, at $f_0 = 3$ GHz. Determine the percent bandwidth for which the SWR ≤ 1.5.

Solution

From (6.25), the characteristic impedance of the matching section is

$$Z_1 = \sqrt{Z_0 Z_L} = \sqrt{(50)(10)} = 22.36\ \Omega,$$

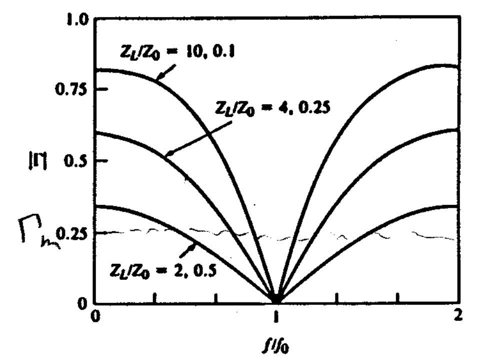


FIGURE 6.12 Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.

and the length of the matching section is $\lambda/4$ at 3 GHz. An SWR of 1.5 corresponds to a reflection coefficient magnitude of

$$\Gamma_m = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{1.5 - 1}{1.5 + 1} = 0.2.$$

The fractional bandwidth is computed from (6.33) as

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right]$$

$$= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{0.2}{\sqrt{1-(0.2)^2}}\frac{2\sqrt{(50)(10)}}{|10 - 50|}\right]$$

$$= 0.29, \text{ or } 29\%.$$

6.5 THE THEORY OF SMALL REFLECTIONS

The quarter-wave transformer provides a simple means of matching any real load impedance to any line impedance. For applications requiring more bandwidth than a single quarter-wave section can provide, multisection transformers can be used. The design of such transformers is the subject of the next two sections, but prior to that material we need to derive some approximate results for the total reflection coefficient caused by the partial reflections from several small discontinuities. This topic is generally referred to as the theory of small reflections [1].

Single-Section Transformer

Consider the single-section transformer shown in Figure 6.13; we will derive an approximate expression for the overall reflection coefficient Γ. The partial reflection and

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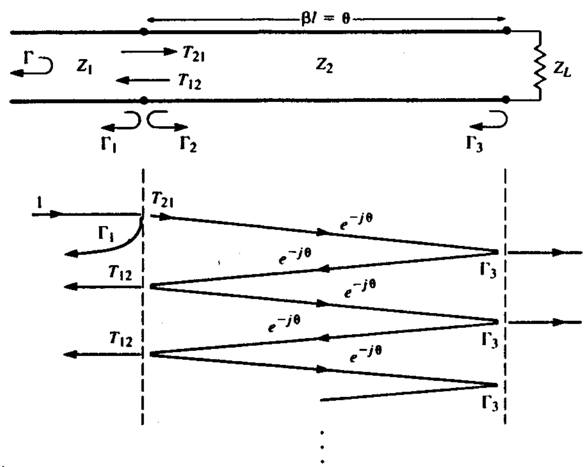


FIGURE 6.13 Partial reflections and transmissions on a single-section matching transformer.

transmission coefficients are

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \qquad 6.34$$

$$\Gamma_2 = -\Gamma_1, \qquad 6.35$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}, \qquad 6.36$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}, \qquad 6.37$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}. \qquad 6.38$$

We can compute the total reflection, Γ, seen by the feed line by the impedance method or by the multiple reflection method, as discussed in Section 3.5. For our present purpose the latter technique is preferred, so we can express the total reflection as an infinite sum of partial reflections and transmissions as follows:

$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-4j\theta} + \cdots$$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-2jn\theta}. \qquad 6.39$$

Using the geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

(6.39) can be expressed in closed form as

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-2j\theta}}{1 - \Gamma_2\Gamma_3 e^{-2j\theta}}. \qquad 6.40$$

From (6.35), (6.37), and (6.38), we use $\Gamma_2 = -\Gamma_1$, $T_{21} = 1 + \Gamma_1$, and $T_{12} = 1 - \Gamma_1$ in (6.40) to give

$$\boxed{\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1\Gamma_3 e^{-2j\theta}}}. \qquad 6.41$$

Now if the discontinuities between the impedances Z_1, Z_2 and Z_2, Z_L are small, then $|\Gamma_1\Gamma_3| \ll 1$, so we can approximate (6.41) as

$$\Gamma \simeq \Gamma_1 + \Gamma_3 e^{-2j\theta}. \qquad 6.42$$

for small discontinuities ($|\Gamma_1\Gamma_3| \ll 1$)

This result states the intuitive idea that the total reflection is dominated by the reflection from the initial discontinuity between Z_1 and Z_2 (Γ_1), and the first reflection from the discontinuity between Z_2 and Z_L ($\Gamma_3 e^{-2j\theta}$). The $e^{-2j\theta}$ term accounts for the phase delay when the incident wave travels up and down the line. The following example demonstrates the accuracy of this approximation.

EXAMPLE 6.6

Consider the quarter-wave transformer of Figure 6.13, with $Z_1 = 100\ \Omega$, $Z_2 = 150\ \Omega$, and $Z_L = 225\ \Omega$. Evaluate the worst-case percent error in computing $|\Gamma|$ from the approximate expression of (6.42).

Solution

The partial reflection coefficients from (6.34) and (6.36) are

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{150 - 100}{150 + 100} = 0.2,$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2} = \frac{225 - 150}{225 + 150} = 0.2.$$

Since the approximate expression for Γ in (6.42) is identical to the numerator for the exact expression in (6.41), the greatest error will occur when the denominator of (6.41) departs from unity to the greatest extent. This occurs for $\theta = 0$ or 180°, since for $\theta = 90°$ both results are zero. Then (6.41) gives the exact result as $\Gamma = 0.384$, while (6.42) gives the approximate result as $\Gamma = 0.4$. The error is about 4%.

greatest error

Multisection Transformer

Now consider the multisection transformer shown in Figure 6.14. This transformer consists of N equal-length (*commensurate*) sections of transmission lines. We will derive an approximate expression for the total reflection coefficient Γ.

Partial reflection coefficients can be defined at each junction, as follows:

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \qquad 6.43a$$

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}, \qquad 6.43b$$

$$\Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}. \qquad 6.43c$$

We also assume that all Z_n increase or decrease monotonically across the transformer, and that Z_L is real. This implies that all Γ_n will be real, and of the same sign ($\Gamma_n > 0$ if $Z_L > Z_0$; $\Gamma_n < 0$ if $Z_L < Z_0$). Then using the results of the previous section, the overall reflection coefficient can be approximated as

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta}. \qquad 6.44$$

N TRLs → N+1 terms

Further assume that the transformer can be made symmetrical, so that $\Gamma_0 = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, $\Gamma_2 = \Gamma_{N-2}$, etc. (Note that this does *not* imply that the Z_ns are symmetrical). Then (6.44) can be written as

$$\Gamma(\theta) = e^{-jN\theta}\{\Gamma_0[e^{jN\theta} + e^{-jN\theta}] + \Gamma_1[e^{j(N-2)\theta} + e^{-j(N-2)\theta}] + \cdots\}. \qquad 6.45$$

If N is odd, the last term is $\Gamma_{(N-1)/2}(e^{j\theta} - e^{-j\theta})$, while if N is even the last term is $\Gamma_{N/2}$. Equation (6.45) is then seen to be of the form of a finite Fourier cosine series in θ, which can be written as

$$\Gamma(\theta) = 2e^{-jN\theta}\Big[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + \frac{1}{2}\Gamma_{N/2}\Big], \quad \text{for } N \text{ even}, \qquad 6.46a$$

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + \Gamma_{(N-1)/2}\cos\theta], \quad \text{for } N \text{ odd}. \qquad 6.46b$$

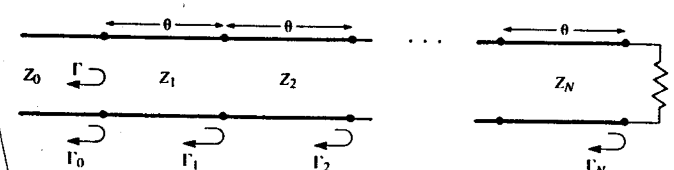


FIGURE 6.14 Partial reflection coefficients for a multisection matching transformer.

The importance of these results lies in the fact that we can synthesize any desired reflection coefficient response as a function of frequency (θ), by properly choosing the Γ_ns and using enough sections (N). This should be clear from the realization that a Fourier series can represent an arbitrary smooth function, if enough terms are used. In the next two sections we will show how to use this theory to design multisection transformers for two of the most commonly used passband responses: the binomial (maximally flat) response, and the Chebyshev (equal ripple) response.

6.6 BINOMIAL MULTISECTION MATCHING TRANSFORMERS (maximally flat)

The passband response of a binomial matching transformer is optimum in the sense that, for a given number of sections, the response is as flat as possible near the design frequency. Thus, such a transformer is also known as maximally flat. This type of response is designed, for an N-section transformer, by setting the first $N-1$ derivatives of $|\Gamma(\theta)|$ to zero, at the center frequency f_0. Such a response can be obtained if we let

$$\boxed{\Gamma(\theta) = A(1 + e^{-2j\theta})^N.} \qquad 6.47$$

Then the magnitude $|\Gamma(\theta)|$ is

$$|\Gamma(\theta)| = |A||e^{-j\theta}|^N|e^{j\theta} + e^{-j\theta}|^N = 2^N|A||\cos\theta|^N \qquad 6.48$$

Note that $|\Gamma(\theta)| = 0$ for $\theta = \pi/2$, and that $(d^n|\Gamma(\theta)|)/d\theta^n = 0$ at $\theta = \pi/2$ for $n = 1, 2, \ldots, N-1$. ($\theta = \pi/2$ corresponds to the center frequency f_0, for which $\ell = \lambda/4$ and $\theta = \beta\ell = \pi/2$.)

We can determine the constant A by letting $f \to 0$. Then $\theta = \beta\ell = 0$, and (6.48) reduces to

$$|\Gamma(0)| = 2^N|A| = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|, \quad = |\Gamma(0)|$$

since for $f = 0$ all sections are of zero electrical length. Thus the constant A can be written as

$$\boxed{A = 2^{-N}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|.} \qquad 6.49$$

Now expand $\Gamma(\theta)$ in (6.47) according to the binomial expansion:

$$\Gamma(\theta) = A(1 + e^{-2j\theta})^N = A\sum_{n=0}^{N} C_n^N e^{-2jn\theta}, \qquad 6.50$$

where

$$C_n^N = \frac{N!}{(N-n)!n!}, \qquad 6.51$$

are the binomial coefficients. Note that $C_n^N = C_{N-n}^N$, $C_0^N = 1$, and $C_1^N = N = C_{N-1}^N$. The key step is now to equate the desired passband response as given in (6.50), to the actual response as given (approximately) by (6.44):

$$\Gamma(\theta) = A\sum_{n=0}^{N} C_n^N e^{-2jn\theta} = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta}.$$

This shows that the Γ_n must be chosen as

$$\boxed{\Gamma_n = AC_n^N.} \qquad 6.52$$

where A is given by (6.49), and C_n^N is a binomial coefficient.

At this point, the characteristic impedances Z_n can be found via (6.43), but a simpler solution can be obtained using the following approximation [1]. Since we assumed that the Γ_n are small, we can write

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \simeq \frac{1}{2}\ln\frac{Z_{n+1}}{Z_n},$$

since $\ln x \simeq 2(x - 1/x + 1)$. Then, using (6.52) and (6.49) gives

Approximation $= \frac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n} \approx \frac{1}{2}\ln\frac{Z_{n+1}}{Z_n}$

$$\ln\frac{Z_{n+1}}{Z_n} = 2\Gamma_n = 2AC_n^N = 2(2^{-N})\frac{Z_L - Z_0}{Z_L + Z_0}C_n^N = 2^{-N}C_n^N \ln\frac{Z_L}{Z_0}, \qquad 6.53$$

which can be used to find Z_{n+1}, starting with $n = 0$. These results are approximate, but generally give usable results for $0.5Z_0 < Z_L < 2Z_0$.

Exact results can be found by using the transmission line equations for each section and numerically solving for the characteristic impedances [2]. The results of such calculations are listed in Table 6.1, which give the exact line impedances for $N = 2, 3, 4, 5$, and 6 section binomial matching transformers, for various ratios of load impedance, Z_L, to feed line impedance, Z_0. The table gives results only for $Z_L/Z_0 > 1$; if $Z_L/Z_0 < 1$, the results for Z_0/Z_L should be used, but with Z_1 starting at the load end. This is because the solution is symmetric about $Z_L/Z_0 = 1$; the same transformer that matches Z_L to Z_0 can be reversed and used to match Z_0 to Z_L. More extensive tables can be found in reference [2].

The bandwidth of the binomial transformer can be evaluated as follows. As in Section 6.4, let Γ_m be the maximum value of reflection coefficient that can be tolerated over the passband. Then from (6.48),

$$\Gamma_m = 2^N|A|\cos^N\theta_m,$$

where $\theta_m < \pi/2$ is the lower edge of the passband, as shown in Figure 6.11. Thus,

$$\theta_m = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{A}\right)^{1/N}\right], \qquad 6.54$$

and using (6.33) gives the fractional bandwidth as

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{A}\right)^{1/N}\right]. \qquad 6.55$$

TABLE 6.1 Binomial Tranformer Design

Z_L/Z_0	N = 2: Z_1/Z_0	Z_2/Z_0	N = 3: Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	N = 4: Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1067	1.3554	1.0520	1.2247	1.4259	1.0257	1.1351	1.3215	1.4624
2.0	1.1892	1.6818	1.0907	1.4142	1.8337	1.0444	1.2421	1.6102	1.9150
3.0	1.3161	2.2795	1.1479	1.7321	2.6135	1.0718	1.4105	2.1269	2.7990
4.0	1.4142	2.8285	1.1907	2.0000	3.3594	1.0919	1.5442	2.5903	3.6633
6.0	1.5651	3.8336	1.2544	2.4495	4.7832	1.1215	1.7553	3.4182	5.3500
8.0	1.6818	4.7568	1.3022	2.8284	6.1434	1.1436	1.9232	4.1597	6.9955
10.0	1.7783	5.6233	1.3409	3.1623	7.4577	1.1613	2.0651	4.8424	8.6110

Z_L/Z_0	N = 5: Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	N = 6: Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_6/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0128	1.0790	1.2247	1.3902	1.4810	1.0064	1.0454	1.1496	1.3048	1.4349	1.4905
2.0	1.0220	1.1391	1.4142	1.7558	1.9569	1.0110	1.0790	1.2693	1.5757	1.8536	1.9782
3.0	1.0354	1.2300	1.7321	2.4390	2.8974	1.0176	1.1288	1.4599	2.0549	2.6577	2.9481
4.0	1.0452	1.2995	2.0000	3.0781	3.8270	1.0225	1.1661	1.6129	2.4800	3.4302	3.9120
6.0	1.0596	1.4055	2.4495	4.2689	5.6625	1.0296	1.2219	1.8573	3.2305	4.9104	5.8275
8.0	1.0703	1.4870	2.8284	5.3800	7.4745	1.0349	1.2640	2.0539	3.8950	6.3291	7.7302
10.0	1.0789	1.5541	3.1623	6.4346	9.2687	1.0392	1.2982	2.2215	4.5015	7.7030	9.6228

EXAMPLE 6.7

Design a three-section binomial transformer to match a 50 Ω load to a 100 Ω line, and calculate the bandwidth for $\Gamma_m = 0.05$. Plot the reflection coefficient magnitude versus normalized frequency for the exact designs using 1, 2, 3, 4, and 5 sections.

Solution
For $N = 3, Z_L = 50\ \Omega, Z_0 = 100\ \Omega$ we have, from (6.49),

$$A = 2^{-N}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = 2^{-3}\left|\frac{50-100}{50+100}\right| = 0.0417.$$

From (6.55) the bandwidth is

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{A}\right)^{1/N}\right] = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{0.5}{0.0417}\right)^{1/3}\right] = 0.71, \text{ or } 71\%.$$

The necessary binomial coefficients are

$$C_0^3 = \frac{3!}{3!0!} = 1,$$
$$C_1^3 = \frac{3!}{2!1!} = 3,$$
$$C_2^3 = \frac{3!}{1!2!} = 3.$$

Then using (6.53) gives the required characteristic impedances as

$$n = 0:\ \ln Z_1 = \ln Z_0 + 2^{-N}C_0^3 \ln\frac{Z_L}{Z_0}$$
$$= \ln 100 + 2^{-3}(1)\ln\frac{50}{100} = 4.518,$$
$$Z_1 = 91.7\ \Omega;$$

$$n = 1:\ \ln Z_2 = \ln Z_1 + 2^{-N}C_1^3 \ln\frac{Z_L}{Z_0}$$
$$= \ln 91.7 + 2^{-3}(3)\ln\frac{50}{100} = 4.26,$$
$$Z_2 = 70.7\ \Omega;$$

$$n = 2:\ \ln Z_3 = \ln Z_2 + 2^{-N}C_2^3 \ln\frac{Z_L}{Z_0}$$
$$= \ln 70.7 + 2^{-3}(3)\ln\frac{50}{100} = 4.00,$$
$$Z_3 = 54.5\ \Omega.$$

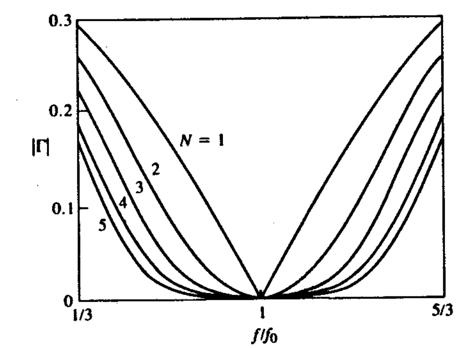


FIGURE 6.15 Reflection coefficient magnitude versus frequency for multisection binomial matching transformers of Example 6.7. $Z_L = 50\ \Omega$ and $Z_0 = 100\ \Omega$.

To use the data in Table 6.1, we reverse the source and load impedances and consider the problem of matching a 100 Ω load to a 50 Ω line. Then $Z_L/Z_0 = 2.0$, and we obtain the exact characteristic impedances as $Z_1 = 91.7\ \Omega, Z_2 = 70.7\ \Omega$, and $Z_3 = 54.5\ \Omega$, which agree with the approximate results to three significant digits. Figure 6.15 shows the reflection coefficient magnitude versus frequency for exact designs using $N = 1, 2, 3, 4$, and 5 sections. Observe that greater bandwidth is obtained for transformers using more sections. ○

6.7 CHEBYSHEV MULTISECTION MATCHING TRANSFORMERS

In contrast with the binomial matching transformer, the Chebyshev transformer optimizes bandwidth at the expense of passband ripple. If such a passband characteristic can be tolerated, the bandwidth of the Chebyshev transformer will be substantially better than that of the binomial transformer, for a given number of sections. The Chebyshev transformer is designed by equating $\Gamma(\theta)$ to a Chebyshev polynomial, which has the optimum characteristics needed for this type of transformer. Thus we will first discuss the properties of the Chebyshev polynomials, and then derive a design procedure for Chebyshev matching transformers using the small reflection theory of Section 6.5.

Chebyshev Polynomials

The nth order Chebyshev polynomial is a polynomial of degree n, and is denoted by $T_n(x)$. The first four Chebyshev polynomials are

$$T_1(x) = x, \qquad 6.56a$$
$$T_2(x) = 2x^2 - 1, \qquad 6.56b$$

$$T_3(x) = 4x^3 - 3x, \tag{6.56c}$$

$$T_4(x) = 8x^4 - 8x^2 + 1. \tag{6.56d}$$

Higher-order polynomials can be found using the following recurrence formula:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x). \tag{6.57}$$

The first four Chebyshev polynomials are plotted in Figure 6.16, from which the following very useful properties of Chebyshev polynomials can be noted:

- For $-1 \le x \le 1$, $|T_n(x)| \le 1$. In this range, the Chebyshev polynomials oscillate between ± 1. This is the equal ripple property, and this region will be mapped to the passband of the matching transformer.
- For $|x| > 1$, $|T_n(x)| > 1$. This region will map to the frequency range outside the passband.
- For $|x| > 1$, the $|T_n(x)|$ increases faster with x as n increases.

Now let $x = \cos\theta$ for $|x| < 1$. Then it can be shown that the Chebyshev polynomials can be expressed as

$$T_n(\cos\theta) = \cos n\theta,$$

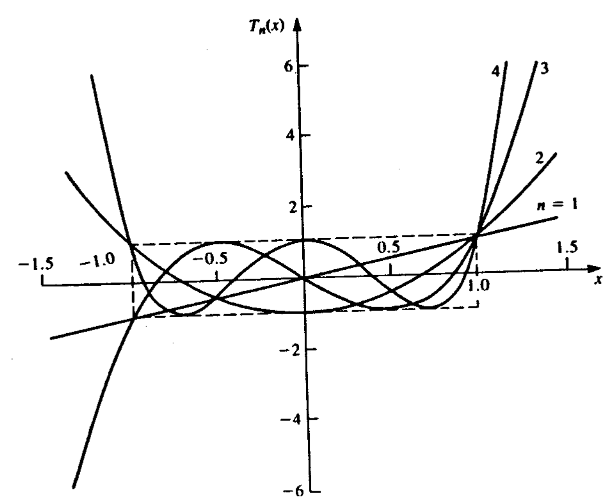


FIGURE 6.16 The first four Chebyshev polynomials, $T_n(x)$.

or more generally as

$$T_n(x) = \cos(n\cos^{-1} x), \quad \text{for } |x| < 1, \tag{6.58a}$$

$$T_n(x) = \cosh(n\cosh^{-1} x), \quad \text{for } |x| > 1. \tag{6.58b}$$

We desire equal ripple in the passband of the transformer, so it is necessary to map θ_m to $x = 1$ and $\pi - \theta_m$ to $x = -1$, where θ_m and $\pi - \theta_m$ are the lower and upper edges of the passband, as shown in Figure 6.11. This can be accomplished by replacing $\cos\theta$ in (6.58a) with $\cos\theta/\cos\theta_m$:

$$T_n\left(\frac{\cos\theta}{\cos\theta_m}\right) = T_n(\sec\theta_m\cos\theta) = \cos n\left[\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_m}\right)\right]. \tag{6.59}$$

Then $|\sec\theta_m\cos\theta| \le 1$ for $\theta_m < \theta < \pi - \theta_m$, so $|T_n(\sec\theta_m\cos\theta)| \le 1$ over this same range.

Since $\cos^n\theta$ can be expanded into a sum of terms of the form $\cos(n - 2m)\theta$, the Chebyshev polynomials of (6.56) can be rewritten in the following useful form:

$$T_1(\sec\theta_m\cos\theta) = \sec\theta_m\cos\theta, \tag{6.60a}$$

$$T_2(\sec\theta_m\cos\theta) = \sec^2\theta_m(1 + \cos 2\theta) - 1, \tag{6.60b}$$

$$T_3(\sec\theta_m\cos\theta) = \sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m\cos\theta, \tag{6.60c}$$

$$T_4(\sec\theta_m\cos\theta) = \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3), \\ - 4\sec^2\theta_m(\cos 2\theta + 1) + 1. \tag{6.60d}$$

The above results can be used to design matching transformers with up to four sections, and will also be used in later chapters for the design of directional couplers and filters.

Design of Chebyshev Transformers

We can now synthesize a Chebyshev equal-ripple passband by making $\Gamma(\theta)$ proportional to $T_N(\sec\theta_m\cos\theta)$, where N is the number of sections in the transformer. Thus, using (6.46),

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0\cos N\theta + \Gamma_1\cos(N-2)\theta + \cdots + \Gamma_n\cos(N-2n)\theta + \cdots] \\ = Ae^{-jN\theta}T_N(\sec\theta_m\cos\theta), \tag{6.61}$$

where the last term in the series of (6.61) is $(1/2)\Gamma_{N/2}$ for N even and $\Gamma_{(N-1)/2}\cos\theta$ for N odd. As in the binomial transformer case, we can find the constant A by letting $\theta = 0$, corresponding to zero frequency. Thus,

$$\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_N(\sec\theta_m),$$

so we have

$$A = \frac{Z_L - Z_0}{Z_L + Z_0}\frac{1}{T_N(\sec\theta_m)}. \tag{6.62}$$

Now if the maximum allowable reflection coefficient magnitude in the passband is Γ_m, then from (6.61) $\Gamma_m = A$, since the maximum value of $T_n(\sec\theta_m \cos\theta)$ in the passband is unity. Then from (6.62) θ_m is determined as

$$T_N(\sec\theta_m) = \frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|,$$

or, using (6.58b),

$$\boxed{\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right].} \qquad 6.63$$

Once θ_m is known, the fractional bandwidth can be calculated from (6.33) as

$$\boxed{\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}.} \qquad 6.64$$

From (6.61), the Γ_n can be determined using the results of (6.60) to expand $T_N(\sec\theta_m \cos\theta)$ and equating similar terms of the form $\cos(N-2n)\theta$. The characteristic impedances Z_n can then be found from (6.43). This procedure will be illustrated in Example 6.8.

The above results are approximate because of the reliance on small reflection theory, but are general enough to design transformers with an arbitrary ripple level, Γ_m. Table 6.2 gives exact results [2] for a few specific values of Γ_m, for $N = 2, 3$, and 4 sections; more extensive tables can be found in reference [2].

EXAMPLE 6.8

Design a three-section Chebyshev transformer to match a 100 Ω load to a 50 Ω line, with $\Gamma_m = 0.05$, using the above theory. Plot the reflection coefficient magnitude versus normalized frequency for exact designs using 1, 2, 3, and 4 sections.

Solution
From (6.61) with $N = 3$,

$$\Gamma(\theta) = 2e^{-j3\theta}[\Gamma_0 \cos 3\theta + \Gamma_1 \cos\theta] = Ae^{-j3\theta}T_3(\sec\theta_m \cos\theta).$$

Then, $A = \Gamma_m = 0.05$, and from (6.63),

$$\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]$$

$$= \cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{1}{0.05}\left|\frac{100-50}{100+50}\right|\right)\right]$$

$$= 1.395,$$

so, $\theta_m = 44.2°$.
Using (6.60c) for T_3 gives

$$2[\Gamma_0 \cos 3\theta + \Gamma_1 \cos\theta] = A\sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3A\sec\theta_m \cos\theta.$$

TABLE 6.2 Chebyshev Transformer Design

	$N = 2$				$N = 3$					
	$\Gamma_m = 0.05$		$\Gamma_m = 0.20$		$\Gamma_m = 0.05$			$\Gamma_m = 0.20$		
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1347	1.3219	1.2247	1.2247	1.1029	1.2247	1.3601	1.2247	1.2247	1.2247
2.0	1.2193	1.6402	1.3161	1.5197	1.1475	1.4142	1.7429	1.2855	1.4142	1.5558
3.0	1.3494	2.2232	1.4565	2.0598	1.2171	1.7321	2.4649	1.3743	1.7321	2.1829
4.0	1.4500	2.7585	1.5651	2.5558	1.2662	2.0000	3.1591	1.4333	2.0000	2.7908
6.0	1.6047	3.7389	1.7321	3.4641	1.3383	2.4495	4.4833	1.5193	2.4495	3.9492
8.0	1.7244	4.6393	1.8612	4.2983	1.3944	2.8284	5.7372	1.5766	2.8284	5.0742
10.0	1.8233	5.4845	1.9680	5.0813	1.4385	3.1623	6.9517	1.6415	3.1623	6.0920

	$N = 4$							
	$\Gamma_m = 0.05$				$\Gamma_m = 0.20$			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0892	1.1742	1.2775	1.3772	1.2247	1.2247	1.2247	1.2247
2.0	1.1201	1.2979	1.5409	1.7855	1.2727	1.3634	1.4669	1.5715
3.0	1.1586	1.4876	2.0167	2.5893	1.4879	1.5819	1.8965	2.0163
4.0	1.1906	1.6414	2.4369	3.3597	1.3692	1.7490	2.2870	2.9214
6.0	1.2290	1.8773	3.1961	4.8820	1.4415	2.0231	2.9657	4.1623
8.0	1.2583	2.0657	3.8728	6.3578	1.4914	2.2428	3.5670	5.3641
10.0	1.2832	2.2268	4.4907	7.7930	1.5163	2.4210	4.1305	6.5950

Equating similar terms in $\cos n\theta$ gives the following results:

$$\cos 3\theta:\ 2\Gamma_0 = A\sec^3\theta_m,$$

$$\Gamma_0 = 0.0678;$$

$$\cos\theta:\ 2\Gamma_1 = 3A(\sec^3\theta_m - \sec\theta_m),$$

$$\Gamma_1 = 0.099.$$

From symmetry we also have that

$$\Gamma_3 = \Gamma_0 = 0.0678,$$

and $$\Gamma_2 = \Gamma_1 = 0.099.$$

Then the characteristic impedances are

$$Z_1 = Z_0\frac{1+\Gamma_0}{1-\Gamma_0} = 50\frac{1+.0678}{1-.0678} = 57.27\ \Omega,$$

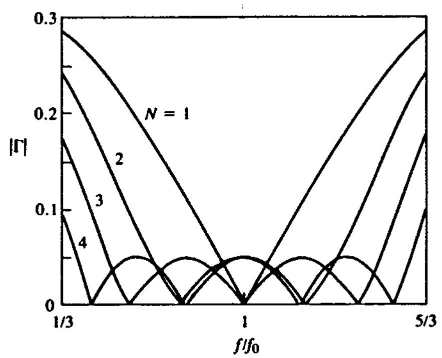


FIGURE 6.17 Reflection coefficient magnitude versus frequency for the multisection matching transformers of Example 6.8.

$$Z_2 = Z_1 \frac{1+\Gamma_1}{1-\Gamma_1} = 57.27 \frac{1+.099}{1-.099} = 69.86\ \Omega,$$

$$Z_3 = Z_L \frac{1-\Gamma_3}{1+\Gamma_3} = 100 \frac{1-.0678}{1+.0678} = 87.30\ \Omega.$$

Note that Z_1 and Z_2 were calculated using (6.43b), starting at the input side of the transformer, but Z_3 was calculated from the load side, using (6.43c). This avoids the cumulative error which would occur if we calculated all impedances in a progression from one side. These values can be compared to the exact values from Table 6.2 of $Z_1 = 57.37\ \Omega$, $Z_2 = 70.71\ \Omega$, and $Z_3 = 87.15\ \Omega$. The bandwidth, from (6.64), is

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi} = 2 - 4\left(\frac{44.2°}{180°}\right) = 1.02,$$

or 102%. This is significantly greater than the bandwidth of the binomial transformer of Example 6.7 (71%), which was for the same type of mismatch. The trade-off, of course, is a nonzero ripple in the passband of the Chebyshev transformer.

Figure 6.17 shows reflection coefficient magnitudes versus frequency for the exact designs from Table 6.2 for $N = 1, 2, 3$, and 4 sections. ○

6.8 TAPERED LINES

In the preceding sections we discussed how an arbitrary real load impedance could be matched to a line over a desired bandwidth by using multisection matching transformers.

As the number, N, of discrete sections increases, the step changes in characteristic impedance between the sections become smaller. Thus, in the limit of an infinite number of sections, we approach a continuously tapered line. In practice, of course, a matching transformer must be of finite length, often no more than a few sections long. But instead of discrete sections, the line can be continuously tapered, as suggested in Figure 6.18a. Then by changing the type of taper, we can obtain different passband characteristics.

In this section we will derive an approximate theory, based on the theory of small reflections, to predict the reflection coefficient response as a function of the impedance taper, $Z(z)$. We will then apply these results to a few common types of tapers.

Consider the continuously tapered line of Figure 6.18a as being made up of a number of incremental sections of length Δz, with an impedance change $\Delta Z(z)$ from one section to the next, as shown in Figure 6.18b. Then the incremental reflection coefficient from the step at z is given by

$$\Delta\Gamma = \frac{(Z+\Delta Z) - Z}{(Z+\Delta Z) + Z} \simeq \frac{\Delta Z}{2Z}. \qquad 6.65$$

In the limit as $\Delta z \to 0$, we have an exact differential:

$$d\Gamma = \frac{dZ}{2Z} = \frac{1}{2}\frac{d(\ln Z/Z_0)}{dz}dz, \qquad 6.66$$

since

$$\frac{d(\ln f(z))}{dz} = \frac{1}{f}\frac{df(z)}{dz}.$$

Then, by using the theory of small reflections, the total reflection coefficient at $z = 0$

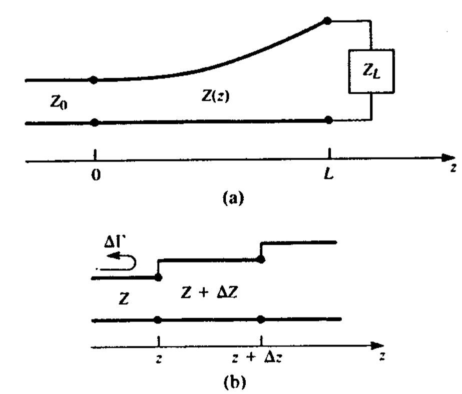


FIGURE 6.18 A tapered transmission line matching section and the model for an incremental length of tapered line. (a) The tapered transmission line matching section. (b) Model for an incremental step change in impedance of the tapered line.

can be found by summing all the partial reflections with their appropriate phase shifts:

total refl. coeff at $z=0$

$$\boxed{\Gamma(\theta) = \frac{1}{2}\int_{z=0}^{L} e^{-2j\beta z}\frac{d}{dz}\ln\left(\frac{Z}{Z_0}\right)dz,} \qquad 6.67$$

where $\theta = 2\beta z$. So if $Z(z)$ is known, $\Gamma(\theta)$ can be found as a function of frequency. Alternatively, if $\Gamma(\theta)$ is specified, then in principal $Z(z)$ can be found. This latter procedure is difficult, and is generally avoided in practice; the reader is referred to references [1], [4] for further discussion along these lines. Here we will consider three special cases of $Z(z)$ impedance tapers, and evaluate the resulting responses.

Exponential Taper

Consider first an exponential taper, where

exponential taper

$$\boxed{Z(z) = Z_0 e^{az}, \quad \text{for } 0 < z < L,} \qquad 6.68$$

as indicated in Figure 6.19a. At $z = 0$, $Z(0) = Z_0$, as desired. At $z = L$, we wish to have $Z(L) = Z_L = Z_0 e^{aL}$, which determines the constant a as

$$a = \frac{1}{L}\ln\left(\frac{Z_L}{Z_0}\right). \qquad 6.69$$

We now find $\Gamma(\theta)$ by using (6.68) and (6.69) in (6.67):

$$\begin{aligned}\Gamma &= \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d}{dz}(\ln e^{az})dz \\ &= \frac{\ln Z_L/Z_0}{2L}\int_0^L e^{-2j\beta z}dz \\ &= \frac{\ln Z_L/Z_0}{2}e^{-j\beta L}\frac{\sin\beta L}{\beta L}.\end{aligned} \qquad 6.70$$

Observe that this derivation assumes that β, the propagation constant of the tapered line, is not a function of z—an assumption which is generally valid only for TEM lines.

The magnitude of the reflection coefficient in (6.70) is sketched in Figure 6.19b; note that the peaks in $|\Gamma|$ decrease with increasing length, as one might expect, and that the length should be greater than $\lambda/2$ ($\beta L > \pi$) to minimize the mismatch at low frequencies.

Triangular Taper

Next consider a triangular taper for $(d \ln Z/Z_0)/dz$, that is,

triangular taper

$$\boxed{Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln Z_L/Z_0} & \text{for } 0 \le z \le L/2 \\ Z_0 e^{(4z/L - 2z^2/L^2 - 1)\ln Z_L/Z_0} & \text{for } L/2 \le z \le L.\end{cases}} \qquad 6.71$$

Then,

$$\frac{d(\ln Z/Z_0)}{dz} = \begin{cases} 4z/L^2 \ln Z_L/Z_0 & \text{for } 0 \le z \le L/2 \\ (4/L - 4z/L^2)\ln Z_L/Z_0 & \text{for } L/2 \le z \le L.\end{cases} \qquad 6.72$$

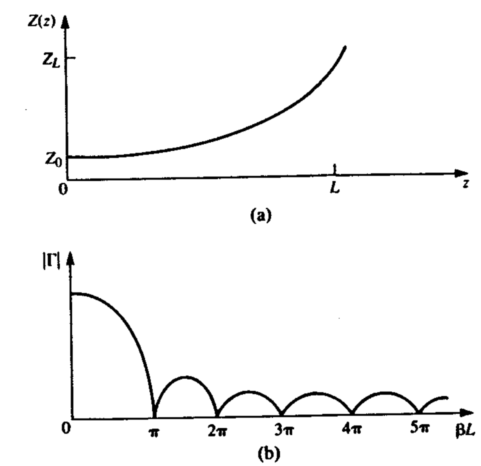


FIGURE 6.19 A matching section with an exponential impedance taper. (a) Variation of impedance. (b) Resulting reflection coefficient magnitude response.

$Z(z)$ is plotted in Figure 6.20a. Evaluating Γ from (6.67) gives

$$\Gamma(\theta) = \frac{1}{2}e^{-j\beta L}\ln\left(\frac{Z_L}{Z_0}\right)\left[\frac{\sin(\beta L/2)}{\beta L/2}\right]^2. \qquad 6.73$$

The magnitude of this result is sketched in Figure 6.20b. Note that, for $\beta L > 2\pi$, the peaks of the triangular taper are lower than the corresponding peaks of the exponential case. But the first null for the triangular taper occurs at $\beta L = 2\pi$, whereas for the exponential taper it occurs at $\beta L = \pi$.

Tri. vs. exponential taper

Klopfenstein Taper

Considering the fact that there are an infinite number of possibilities for choosing an impedance matching taper, it is logical to ask if there is a design which is "best." For a given taper length (greater than a critical value), the Klopfenstein impedance taper [4], [5] has been shown to be optimum in the sense that the reflection coefficient is minimum over the passband. Alternatively, for a maximum reflection coefficient specification in the passband, the Klopfenstein taper yields the shortest matching section.

The Klopfenstein taper is derived from a stepped Chebyshev transformer as the number of sections increases to infinity, and is analogous to the Taylor distribution of antenna array theory. We will not present the details of this derivation, which can be found in references [1], [4]; only the necessary results for the design of Klopfenstein tapers are given below.

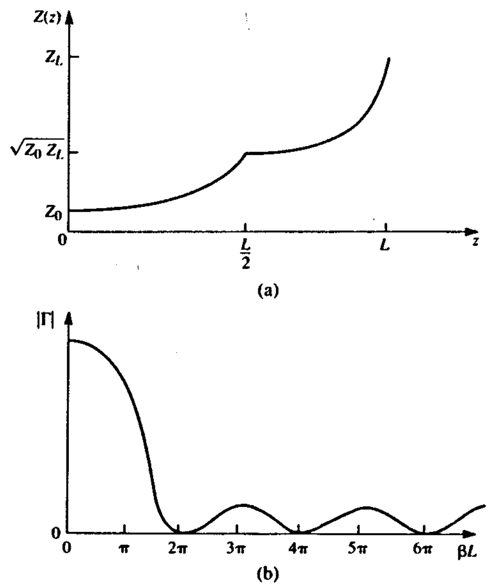


FIGURE 6.20 A matching section with a triangular taper for $d(\ln Z/Z_0)/dz$. (a) Variation of impedance. (b) Resulting reflection coefficient magnitude response.

The logarithm of the characteristic impedance variation for the Klopfenstein taper is given by

$$\boxed{\ln Z(z) = \frac{1}{2}\ln Z_0 Z_L + \frac{\Gamma_0}{\cosh A} A^2 \phi(2z/L - 1, A), \quad \text{for } 0 \le z \le L,} \qquad 6.74$$

where the function $\phi(x, A)$ is defined as

$$\phi(x, A) = -\phi(-x, A) = \int_0^x \frac{I_1(A\sqrt{1-y^2})}{A\sqrt{1-y^2}}\,dy, \quad \text{for } |x| \le 1, \qquad 6.75$$

where $I_1(x)$ is the modified Bessel function. This function takes the following special values:

$$\phi(0, A) = 0$$

$$\phi(x, 0) = \frac{x}{2}$$

$$\phi(1, A) = \frac{\cosh A - 1}{A^2},$$

but otherwise must be calculated numerically. A very simple and efficient method for doing this is available [6].

The resulting reflection coefficient is given by

$$\Gamma(\theta) = \Gamma_0 e^{-j\beta L}\frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A}, \qquad \text{for } \beta L > A. \qquad 6.76$$

If $\beta L < A$, the $\cos\sqrt{(\beta L)^2 - A^2}$ term becomes $\cosh\sqrt{A^2 - (\beta L)^2}$.

In (6.74) and (6.76), Γ_0 is the reflection coefficient at zero frequency, given as

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \simeq \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right). \qquad 6.77$$

The passband is defined as $\boxed{\beta L \ge A}$, and so the maximum ripple in the passband is

$$\boxed{\Gamma_m = \frac{\Gamma_0}{\cosh A},} \text{ max. ripple} \quad (\beta L \ge A) \qquad 6.78$$

because $\Gamma(\theta)$ oscillates between $\pm\Gamma_0/\cosh A$ for $\beta L > A$.

It is interesting to note that the impedance taper of (6.74) has steps at $z = 0$ and L (the ends of the tapered section), and so does not smoothly join the source and load impedances. A typical Klopfenstein impedance taper and its response are given in the ~~following~~ example.

Klopfens taper steps $z=0$, ?

EXAMPLE 6.9

Design a triangular taper, an exponential taper, and a Klopfenstein taper (with $\Gamma_m = 0.02$) to match a 50 Ω load to a 100 Ω line. Plot the impedance variations and resulting reflection coefficient magnitudes versus βL.

Solution

Triangular taper: From (6.71) the impedance variation is

$$Z(z) = Z_0 \begin{cases} e^{2(z/L)^2 \ln Z_L/Z_0} & \text{for } 0 \le z \le L/2 \\ e^{(4z/L - 2z^2/L^2 - 1)\ln Z_L/Z_0} & \text{for } L/2 \le z \le L, \end{cases}$$

with $Z_0 = 100$ Ω and $Z_L = 50$ Ω. The resulting reflection coefficient response is given by (6.73):

$$|\Gamma(\theta)| = \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right)\left[\frac{\sin(\beta L/2)}{\beta L/2}\right]^2.$$

Exponential taper: From (6.68) the impedance variation is

$$Z(z) = Z_0 e^{az}, \qquad \text{for } 0 < z < L,$$

with $a = (1/L)\ln Z_L/Z_0 = 0.693/L$. The reflection coefficient response is, from (6.70),

$$|\Gamma(\theta)| = \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right)\frac{\sin\beta L}{\beta L}.$$

Klopfenstein taper: Using (6.77) gives Γ_0 as

$$\Gamma_0 = \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right) = 0.346,$$

and (6.78) gives A as

$$A = \cosh^{-1}\left(\frac{\Gamma_0}{\Gamma_m}\right) = \cosh^{-1}\left(\frac{0.346}{0.002}\right) = 3.543.$$

The impedance taper must be numerically evaluated from (6.74). The reflection coefficient magnitude is given by (6.76):

$$|\Gamma(\theta)| = \Gamma_0 \frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A}.$$

The passband for the Klopfenstein taper is defined as $\beta L > A = 3.543 = 1.13\pi$.

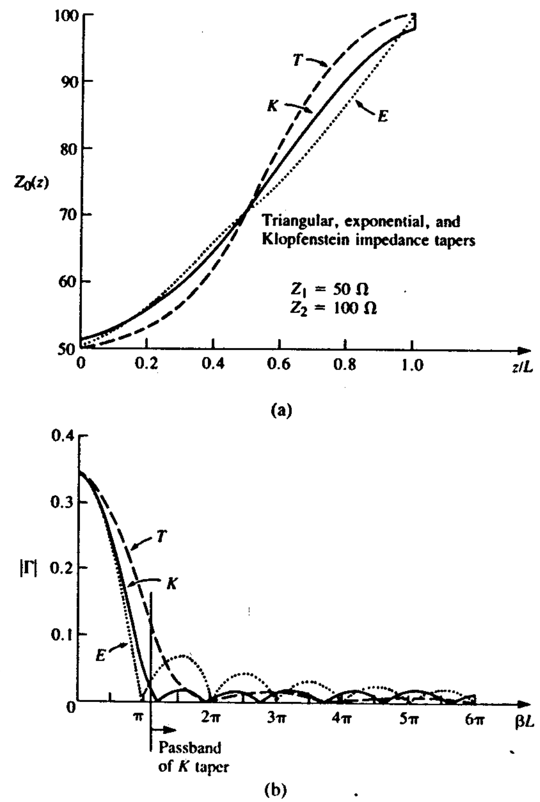


FIGURE 6.21 Solution to Example 6.9. (a) Impedance variations for the triangular, exponential, and Klopfenstein tapers. (b) Resulting reflection coefficient magnitude versus frequency for the tapers of (a).

Klopfenstein gives the shortest length for a specific Γ_m.
It has equal-ripple lobes vs. frequency in its passband

Figure 6.21a,b shows the impedance variations (versus z/L), and the resulting reflection coefficient magnitude (versus βL) for the three types of tapers. The Klopfenstein taper is seen to give the desired response of $|\Gamma| \leq \Gamma_m = 0.02$ for $\beta L \geq 1.13\pi$, which is lower than either the triangular or exponential taper responses. Also note that, like the stepped-Chebyshev matching transformer, the response of the Klopfenstein taper has equal-ripple lobes versus frequency in its passband.

6.9 THE BODE-FANO CRITERIA

In this chapter we discussed several techniques for matching an arbitrary load at a single frequency, using lumped elements, tuning stubs, and single-section quarter-wave transformers. We then presented multisection matching transformers and tapered lines as a means of obtaining broader bandwidths, with various passband characteristics. We will now close our study of impedance matching with a somewhat qualitative discussion of the theoretical limits that constrain the performance of an impedance matching network.

We limit our discussion to the circuit of Figure 6.1, where a lossless network is used to match an arbitrary complex load, generally over a nonzero bandwidth. From a very general perspective, we might raise the following questions in regard to this problem:

- Can we achieve a perfect match (zero reflection) over a specified bandwidth?
- If not, how good can we do? What is the trade-off between Γ_m, the maximum allowable reflection in the passband, and the bandwidth?
- How complex must the matching network be for a given specification?

These questions can be answered by the Bode-Fano criteria [7], [8] which gives, for certain canonical types of load impedances, a theoretical limit on the minimum reflection coefficient magnitude that can be obtained with an arbitrary matching network. The Bode-Fano criteria thus represents the optimum result that can be ideally achieved, even though such a result may only be approximated in practice. Such optimal results are always important, however, because they give us the upper limit of performance, and provide a benchmark against which a practical design can be compared.

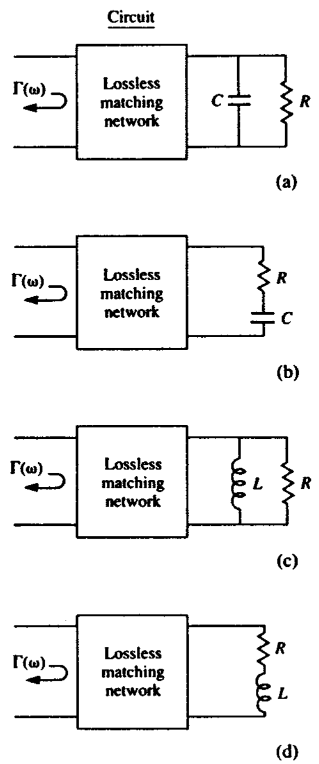
Figure 6.22a shows a lossless network used to match a parallel RC load impedance. The Bode-Fano criteria states that

$$\int_0^\infty \ln\frac{1}{|\Gamma(\omega)|}d\omega \leq \frac{\pi}{RC}, \qquad 6.79$$

where $\Gamma(\omega)$ is the reflection coefficient seen looking into the arbitrary lossless matching network. The derivation of this result is beyond the scope of this text (the interested reader is referred to references [7] and [8]), but our goal here is to discuss the implications of the above result.

Assume that we desire to synthesize a matching network with a reflection coefficient response like that shown in Figure 6.23a. Applying (6.79) to this function gives

$$\int_0^\infty \ln\frac{1}{|\Gamma|}d\omega = \int_{\Delta\omega} \ln\frac{1}{\Gamma_m}d\omega = \Delta\omega \ln\frac{1}{\Gamma_m} \leq \frac{\pi}{RC}, \qquad 6.80$$



Bode-Fano limit

(a) $\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|}\, d\omega < \frac{\pi}{RC}$

(b) $\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|}\, d\omega < \pi\omega_0^2 RC$

(c) $\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|}\, d\omega < \frac{\pi\omega_0^2 L}{R}$

(d) $\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|}\, d\omega < \frac{\pi R}{L}$

FIGURE 6.22 The Bode-Fano limits for RC and RL loads matched with passive and lossless networks (ω_0 is the center frequency of the matching bandwidth). (a) Parallel RC. (b) Series RC. (c) Parallel RL. (d) Series RL.

which leads to the following conclusions:

- For a given load (fixed RC product), a broader bandwidth ($\Delta\omega$) can only be achieved at the expense of a higher reflection coefficient in the passband (Γ_m).
- The passband reflection coefficient Γ_m cannot be zero unless $\Delta\omega = 0$. Thus a perfect match can only be achieved at a finite number of frequencies, as illustrated in Figure 6.23b.
- As R and/or C increase, the quality of the match ($\Delta\omega$ and/or $1/\Gamma_m$) must decrease. Thus, higher-Q circuits are intrinsically harder to match than are lower-Q circuits.

Since $\ln 1/|\Gamma|$ is proportional to the return loss (in dB) at the input of the matching network, (6.79) can be interpreted as requiring that the area between the return loss curve

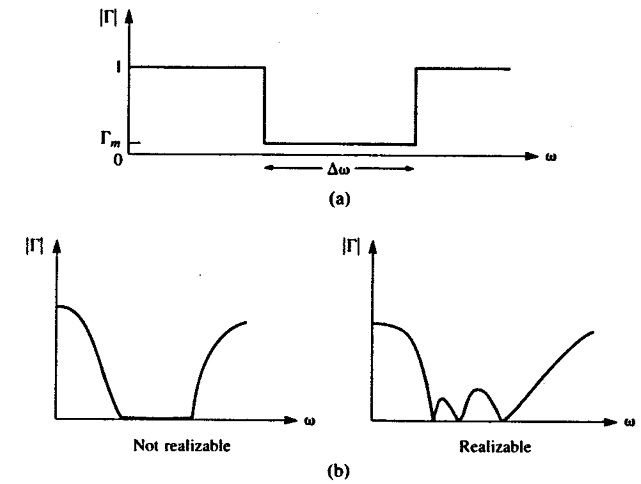


FIGURE 6.23 Illustrating the Bode-Fano criteria. (a) A possible reflection coefficient response. (b) Nonrealizable and realizable reflection coefficient responses.

and the $|\Gamma| = 1$ ($RL = 0$ dB) axis must be less than or equal to a constant. Optimization then implies that the return loss curve be adjusted so that $|\Gamma| = \Gamma_m$ over the passband and $|\Gamma| = 1$ elsewhere, as in Figure 6.23a. In this way, no area under the return loss curve is wasted outside the passband, or lost in regions within the passband for which $|\Gamma| < \Gamma_m$. The square-shaped response of Figure 6.23a is thus the optimum response, but cannot be realized in practice because it would require an infinite number of elements in the matching network. It can be approximated, however, with a reasonably small number of elements, as described in reference [8]. Finally, note that the Chebyshev matching transformer can be considered as a close approximation to the ideal passband of Figure 6.23a, when the ripple of the Chebyshev response is made equal to Γ_m. Figure 6.22 lists the Bode-Fano limits for other types of RC and RL loads.

REFERENCES

[1] R. E. Collin, *Foundations for Microwave Engineering*, McGraw-Hill, N.Y. 1966.
[2] G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures*, Artech House Books, Dedham, Mass. 1980.
[3] P. Bhartia and I. J. Bahl, *Millimeter Wave Engineering and Applications*, Wiley Interscience, N.Y., 1984.
[4] R. E. Collin, "The Optimum Tapered Transmission Line Matching Section," *Proc. IRE*, vol. 44, pp. 539-548, April 1956.
[5] R. W. Klopfenstein, "A Transmission Line Taper of Improved Design," *Proc. IRE*, vol. 44, pp. 31-15, January 1956.

EE3065

MEMORANDUM

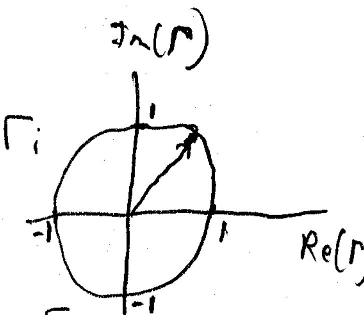
LECTURE 5

Date:

Smith Chart

To:
From:
Subject:

Refl. Coefficient: $\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$



where: $\Gamma_r = |\Gamma| \cos\theta_r$, $\Gamma_i = |\Gamma| \sin\theta_r$ $\quad |\Gamma| \le 1$

The Smith chart lies in the complex plane of Γ

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{\tilde{Z}_L - 1}{\tilde{Z}_L + 1} \rightarrow \tilde{Z}_L = \frac{1+\Gamma}{1-\Gamma}$$

e.g. $\Gamma_A = 0.3 + j0.4$ $\iff$ $\begin{cases} |\Gamma_A| = \sqrt{0.3^2 + 0.4^2} = 0.5 \\ \theta_r = \tan^{-1}(0.4/0.3) = 53^\circ \end{cases}$

↑ horiz. axis ↑ vert. axis

$\Gamma_B = -0.5 - j0.2$ $\iff$ $\begin{cases} |\Gamma_B| = 0.54 \\ \theta_r = 202^\circ \text{ (or } -360^\circ + 202^\circ = -158^\circ) \end{cases}$

(when both Γ_r and Γ_i are negative numbers, θ_r is in the third quadrant in the Γ_r, Γ_i plane)

The unit circle corresponds to $|\Gamma| = 1$. Because $|\Gamma| \le 1$ for a trans line, only that part of the $\Gamma_r - \Gamma_i$ plane that lies within the unit circle has physical meaning.

Impedances on a Smith chart are represented by normalized values, with Z_0, the characteristic impedance of the line, serving as the normalization constant.

$$\tilde{z}_L = Z_L / Z_0$$

Refl. Coef

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{\tilde{Z}_L - 1}{\tilde{Z}_L + 1}$$

$$\rightsquigarrow \quad \tilde{Z}_L = \frac{1+\Gamma}{1-\Gamma}$$

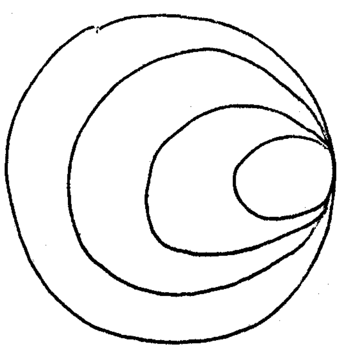
Expressing: $\tilde{Z}_L = r_L + jx_L$, $\Gamma = \Gamma_r + j\Gamma_i$

$$\Downarrow$$

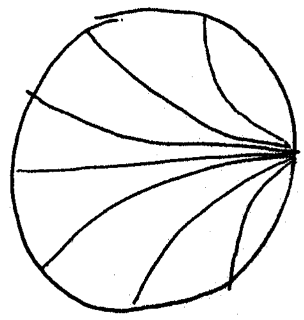
$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}, \quad x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$(r_L, x_L) \xleftrightarrow{\text{unique}} (\Gamma_r, \Gamma_i)$$

If we fix r_L, many possible combinations of values can be assigned to Γ_r, Γ_i, each of which give the same value of r_L $\rightsquigarrow$ All of them are on a constant r_L circle. (for $r_L = 0 \rightarrow |\Gamma| = 1$, cross $(\Gamma_r, \Gamma_i) = (1, 0)$)



Constant r_L circles



Constant x_L circles (arcs)

A given point on the Smith chart, such as $\tilde{Z}_L = 2 - j1$ represents an equivalent reflection coefficient (here. $0.45 \angle -26.6^\circ$). The magnitude $|\Gamma| = 0.45$ is obtained by dividing the length of the line between the center of the Smith chart and the point P by the length of the line between the center of the Smith chart and the edge of the unit circle (the radius of the unit circle corresponds to $|\Gamma|=1$).

The innermost scale of the perimeter of the Smith chart is labeled angle of reflection coefficient in degrees.

$\longrightarrow$ Input Impedance

$$Z_{in} = Z_0 \left[\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}} \right]$$

$$\downarrow$$

Normalized $\quad \tilde{z}_{in} = \dfrac{Z_{in}}{Z_0} = \dfrac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}$

$\Gamma = |\Gamma| e^{j\theta_r}$ is the voltage refl. coef. at the load

$\Gamma_l = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta l)}$ is the phase-shifted voltage refl. coefficient

$\downarrow$

Same magnitude as Γ, but the phase is shifted by $2\beta l$ relative to that of Γ.

$\left(\tilde{z}_{in} = \dfrac{1+\Gamma_l}{1-\Gamma_l} \right)$

To transform $\Gamma \to \Gamma_l$ (or $\tilde{z}_L \to \tilde{z}_{in}$)

(1) Identify position of $\tilde{z}_L$ on Smith chart (Point A)
(2) Draw a circle through A with the center of the circle being at the center of the Smith chart (constant $|\Gamma|$)
(3) Move clockwise on the circle adding to the phase of point A (intersection of radius to A and outer circle — outermost label (wavelengths toward generator))

the distance of the position of $\tilde{z}_{in}$.

(4) Denormalize $\tilde{z}_{in}$ by multiplying with Z_0.

(Distances larger than 0.5λ or smaller than 0 have to be shifted to this interval by adding / subtracting half wavelengths)

e.g. $Z_L = (100 - j50)\ \Omega$, $Z_0 = 50\ \Omega$

$$\tilde{z}_L = Z_L / Z_0 = 2 - j1 \quad (0.287\lambda \text{ on WTG scale})$$

$\tilde{z}_{in}(z = 0.1\lambda)$ → Add $0.287 + 0.1 = 0.387\lambda$

→ new point at the constant Γ circle

→ $\tilde{z}_{in} = 0.6 - j0.66$ → $Z_{in} = \tilde{z}_{in} Z_0 = (30 - j33)\,\Omega$

Standing Wave Ratio

The constant-Γ circle intersects the real axis (Γ_r) at two points, P_{max} and P_{min}. At both points $\Gamma_i = 0$, $\Gamma = \Gamma_r$, $x_L = 0$

From definition of Γ: $\Gamma = \dfrac{\tilde{z}_L - 1}{\tilde{z}_L + 1}$,

P_{min}, P_{max} correspond to: $\Gamma = \Gamma_r = \dfrac{r_L - 1}{r_L + 1}$ (for $\Gamma_i = 0$)

(P_{min} when $r_L < 1$, P_{max} when $r_L > 1$)

$|\Gamma| = \dfrac{S-1}{S+1}$, $S = SWR$. For P_{max}, P_{min} → $|\Gamma| = \Gamma_r$

$$\Rightarrow \Gamma_r = \frac{S-1}{S+1}$$

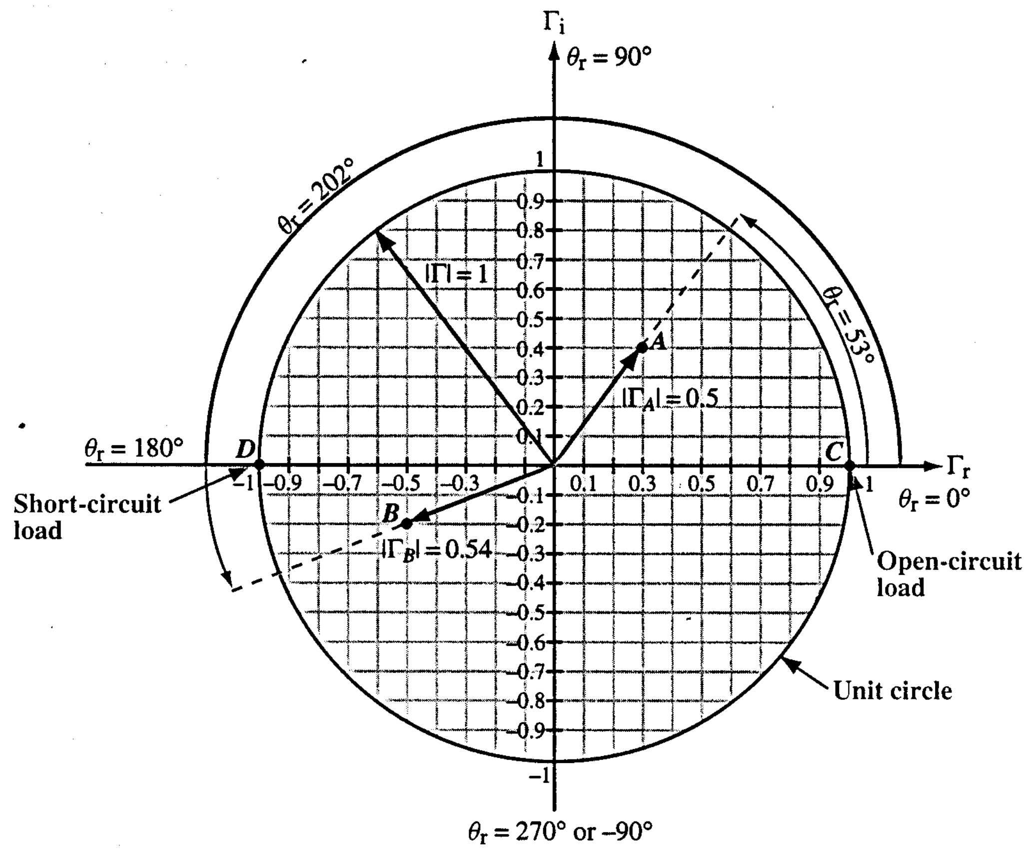


Figure 2-20: The complex Γ plane. Point A is at $\Gamma_A = 0.3 + j0.4 = 0.5e^{j53°}$, and point B is at $\Gamma_B = -0.5 - j0.2 = |0.54|e^{j202°}$. The unit circle corresponds to $|\Gamma| = 1$. At point C, $\Gamma = 1$, corresponding to an open-circuit load, and at point D, $\Gamma = -1$, corresponding to a short circuit.

and

$$\theta_r = \tan^{-1}(0.4/0.3) = 53°.$$

Similarly, point B represents $\Gamma_B = -0.5 - j0.2$, or $|\Gamma_B| = 0.54$ and $\theta_r = 202°$ [or, equivalently, $\theta_r = (360° - 202°) = -158°$]. Note that when both Γ_r and Γ_i are negative numbers θ_r is in the third quadrant in the Γ_r–Γ_i plane. Thus, when using $\theta = \tan^{-1}(\Gamma_i/\Gamma_r)$ to compute θ_r, it may be necessary to add or subtract 180° to obtain the correct value of θ_r.

The *unit circle* shown in Fig. 2-20 corresponds to $|\Gamma| = 1$. Because $|\Gamma| \le 1$ for a transmission line, only that part of the Γ_r–Γ_i plane that lies within the unit circle has physical meaning; hence, future drawings will be limited to the domain contained within the unit circle.

Impedances on a Smith chart are represented by

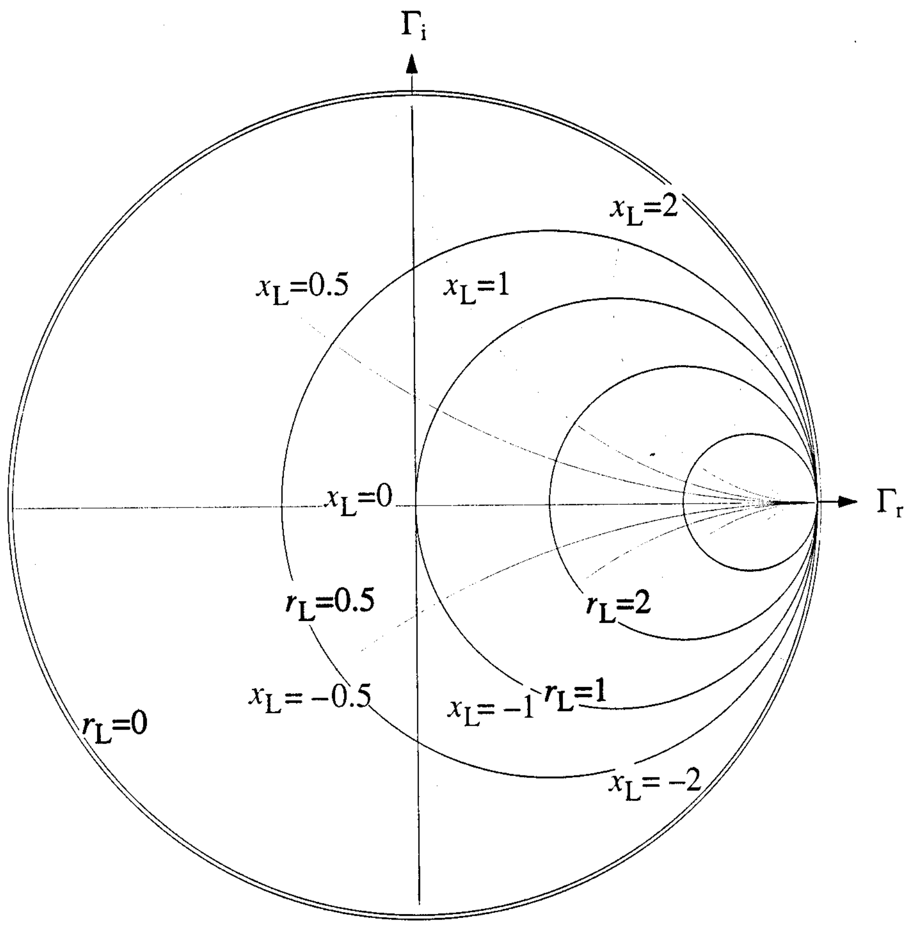


Figure 2-21: Families of r_L and x_L circles within the domain $|\Gamma| \leq 1$.

meaningless). Hence, Eq. (2.100) can generate two families of circles, one family corresponding to positive values of x_L and another corresponding to negative values of x_L. Furthermore, as shown in Fig. 2-21, only part of a given circle falls within the bounds of the unit circle. The families of circles of the two parametric equations given by Eqs. (2.98) and (2.100) plotted for selected values of r_L and x_L constitute the Smith chart shown in Fig. 2-22. A given point on the Smith chart, such as point P in Fig. 2-22, represents a normalized load impedance $z_L = 2 - j1$, with a corresponding voltage reflection coefficient $\Gamma = 0.45 \exp(-j26.6°)$. The magnitude $|\Gamma| = 0.45$ is obtained by dividing the length of the line between the center of the Smith chart and the point P by

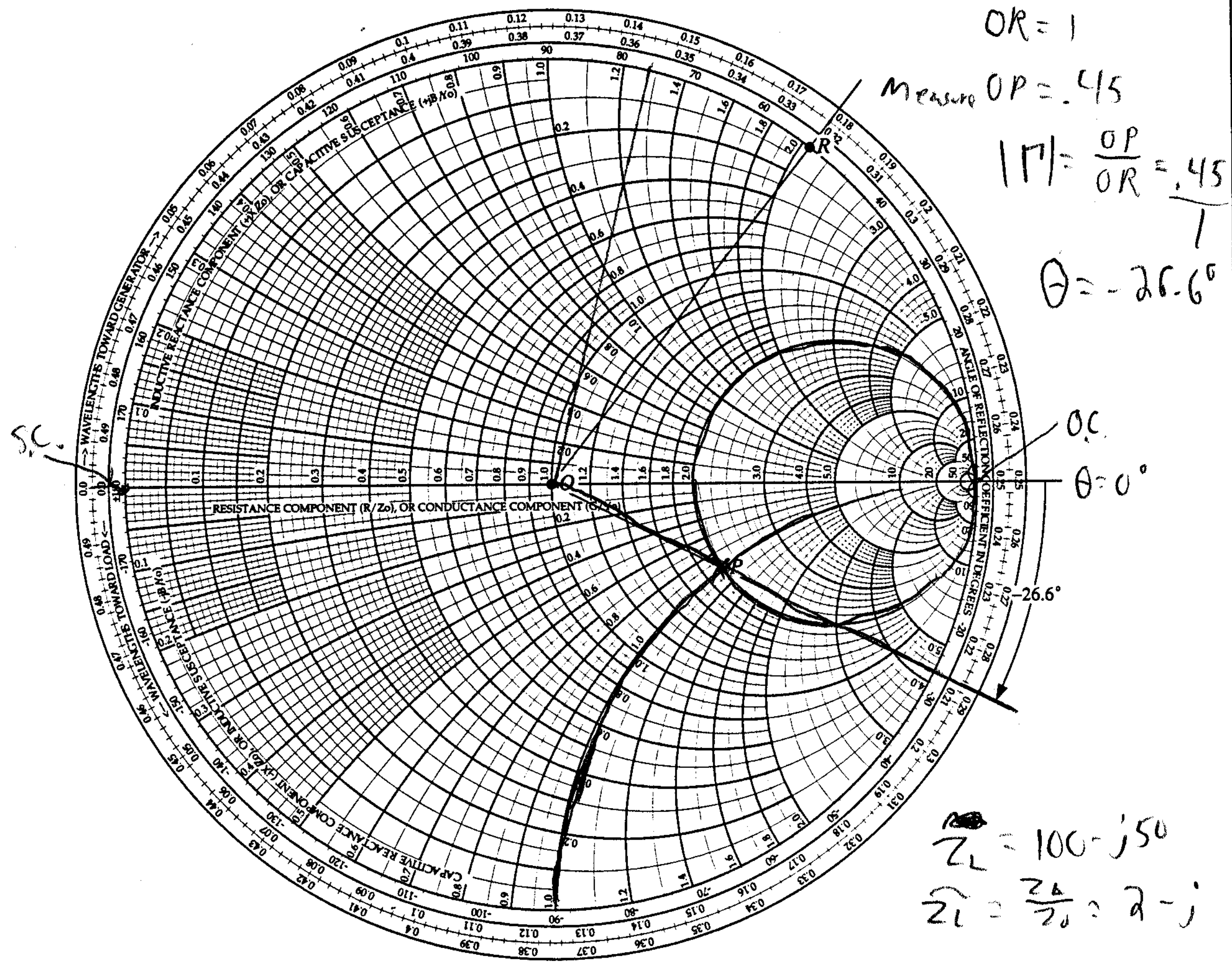


Figure 2-22: Point P represents a normalized load impedance $z_L = 2 - j1$. The reflection coefficient has a magnitude $|\Gamma| = \overline{OP}/\overline{OR} = 0.45$ and an angle $\theta_r = -26.6°$. Point R is an arbitrary point on the $r_L = 0$ circle (which also is the $|\Gamma| = 1$ circle).

the length of the line between the center of the Smith chart and the edge of the unit circle (the radius of the unit circle corresponds to $|\Gamma| = 1$). The perimeter of the Smith chart contains three concentric scales. The innermost scale is labeled *angle of reflection coefficient in degrees*. This is the scale for θ_r. As indicated in Fig. 2-22, $\theta_r = -26.6°$ for point P. The meanings and uses of the other two scales are discussed next.

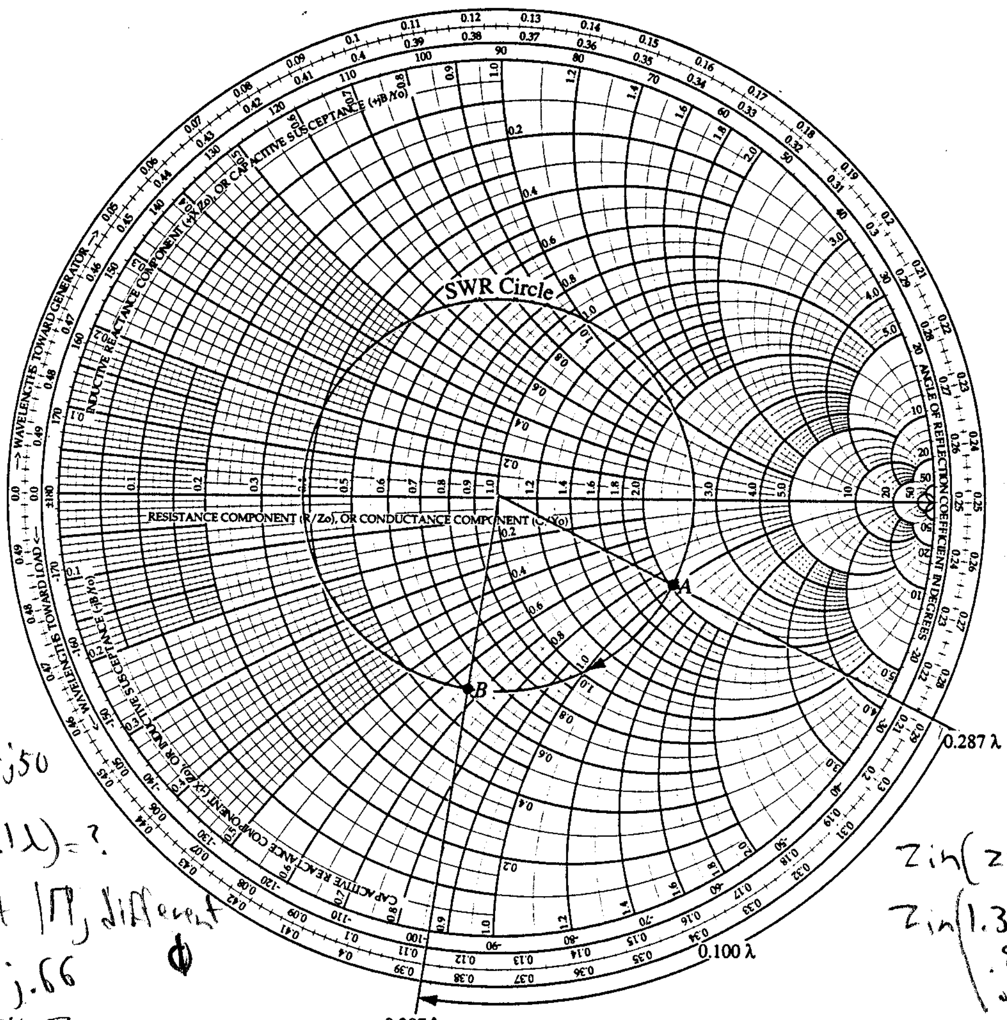
$Z_L = 100 - j50$

$Z_{in}(z = -.1\lambda) = ?$

constant $|\Gamma|$, different ϕ

$\tilde{Z}_{in} = .6 - j.66$

$Z_{in} = \tilde{Z}_{in} Z_0$



s.c. $[0, .5\lambda]$

$Z_{in}(z = -1.1\lambda) = ?$

$Z_{in}(1.387\lambda + (n\lambda/2))$

.887λ

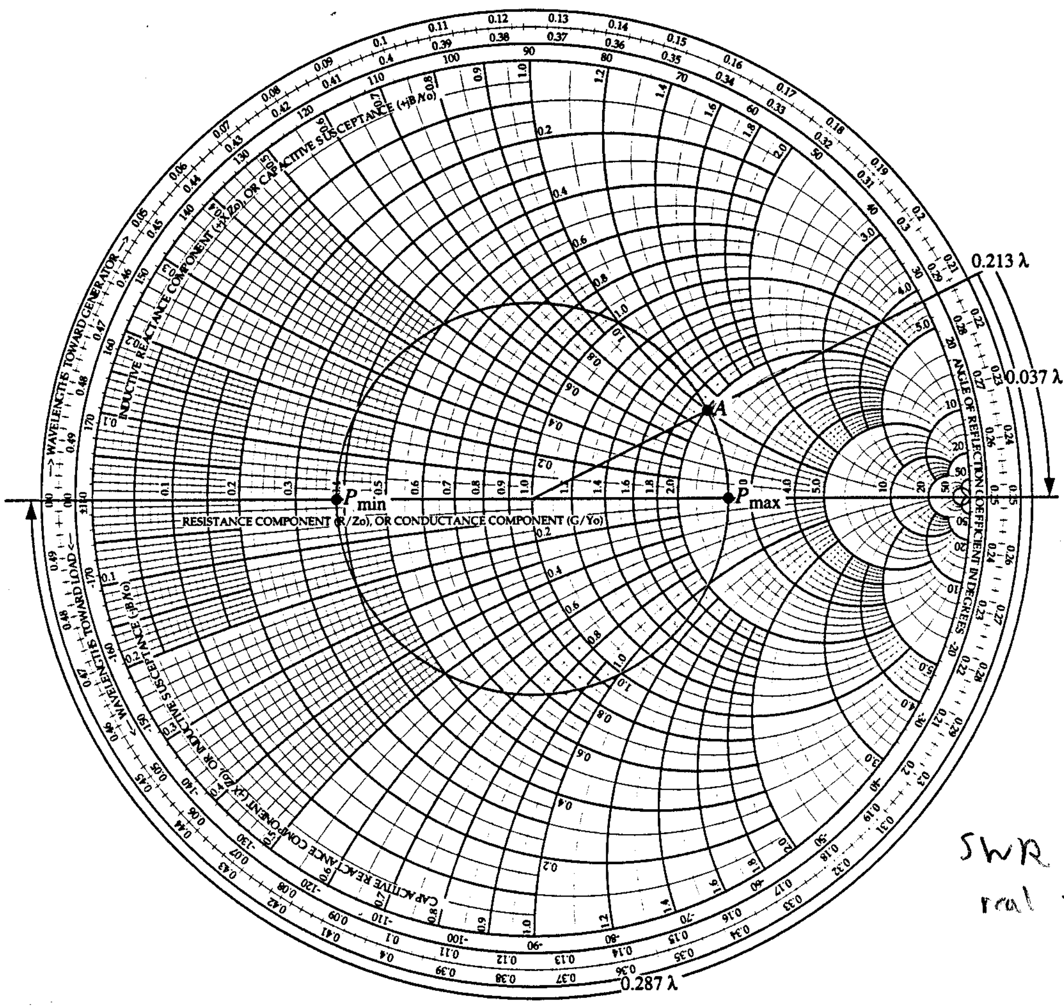
.387λ all equal

Figure 2-23: Point A represents a normalized load $z_L = 2 - j1$ at 0.287λ on the WTG scale. Point B represents the line input at 0.1λ from the load. At B, $z_{in} = 0.6 - j0.66$.

voltage standing-wave ratio (SWR) is related to $|\Gamma|$ by Eq. (2.59) as

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}. \qquad (2.107)$$

Thus, a constant value of $|\Gamma|$ corresponds to a specific value for S. As was stated earlier, to transform z_L to z_{in}, we need to maintain $|\Gamma|$ constant, which means staying on the SWR circle, and to decrease the phase of Γ by $2\beta l$. This is equivalent to moving a distance $l = 0.1\lambda$ toward



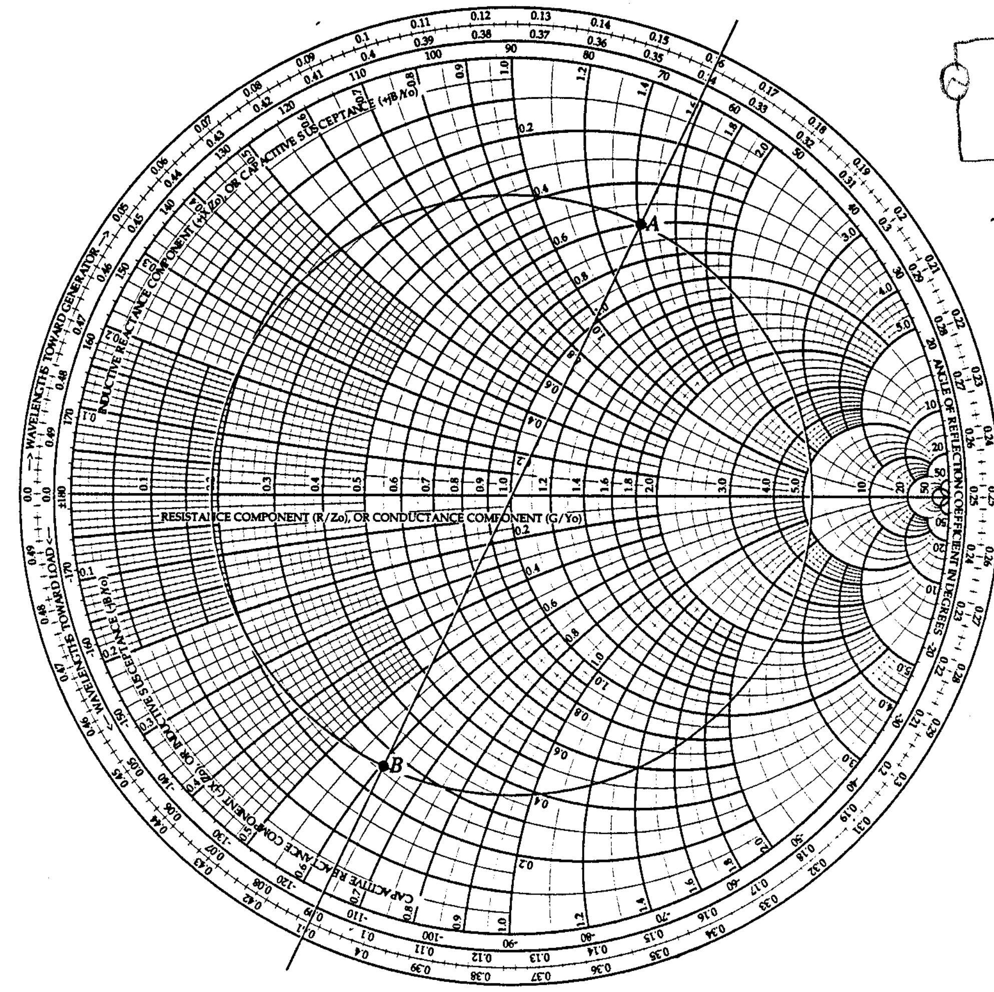
SWR always real ∈ [1, ∞)

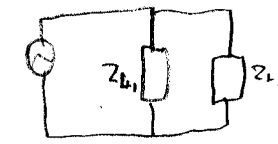
Figure 2-24: Point A represents a normalized load with $z_L = \overset{A}{\underbrace{2 + j1}}$. The standing wave ratio is $S = 2.6$ (at P_{max}), the distance between the load and the first voltage maximum is $l_{max} = (0.25 - 0.213)\lambda = 0.037\lambda$, and the distance between the load and the first voltage minimum is $l_{min} = (0.037 + 0.25)\lambda = 0.287\lambda$.

2-9.4 Impedance to Admittance Transformations

In solving certain types of transmission line problems, it is often more convenient to work with admittances than with impedances. Any impedance Z is in general a complex quantity consisting of a resistance R and a reactance X:

$$Z = R + jX \quad (\Omega). \tag{2.112}$$





$Z_L = \dfrac{Z_{L_1} Z_{L_2}}{Z_{L_1} + Z_{L_2}}$

$Y_L = Y_{L_1} + Y_{L_2}$

$Y = \frac{1}{Z}$

Figure 2-25: Point A represents a normalized load $z_L = 0.6 + j1.4$. Its corresponding normalized admittance is $y_L = 0.25 - j0.6$, and it is at point B.

$y_L = Y_L / Y_0 = Y_L / (1/Z_0) = Y_L Z_0$

Solution: (a) The normalized load impedance is

$$z_L = \frac{Z_L}{Z_0} = \frac{25 + j50}{50} = 0.5 + j1,$$

which is marked as point A on the Smith chart in Fig. 2-26. Using a ruler, a radial line is drawn from the center of the chart at point O through point A, outward to the outer perimeter of the chart. The line crosses the scale labeled "angle of reflection coefficient in degrees" at $\theta_r = 83°$. Next, a ruler is used to measure the length d_A

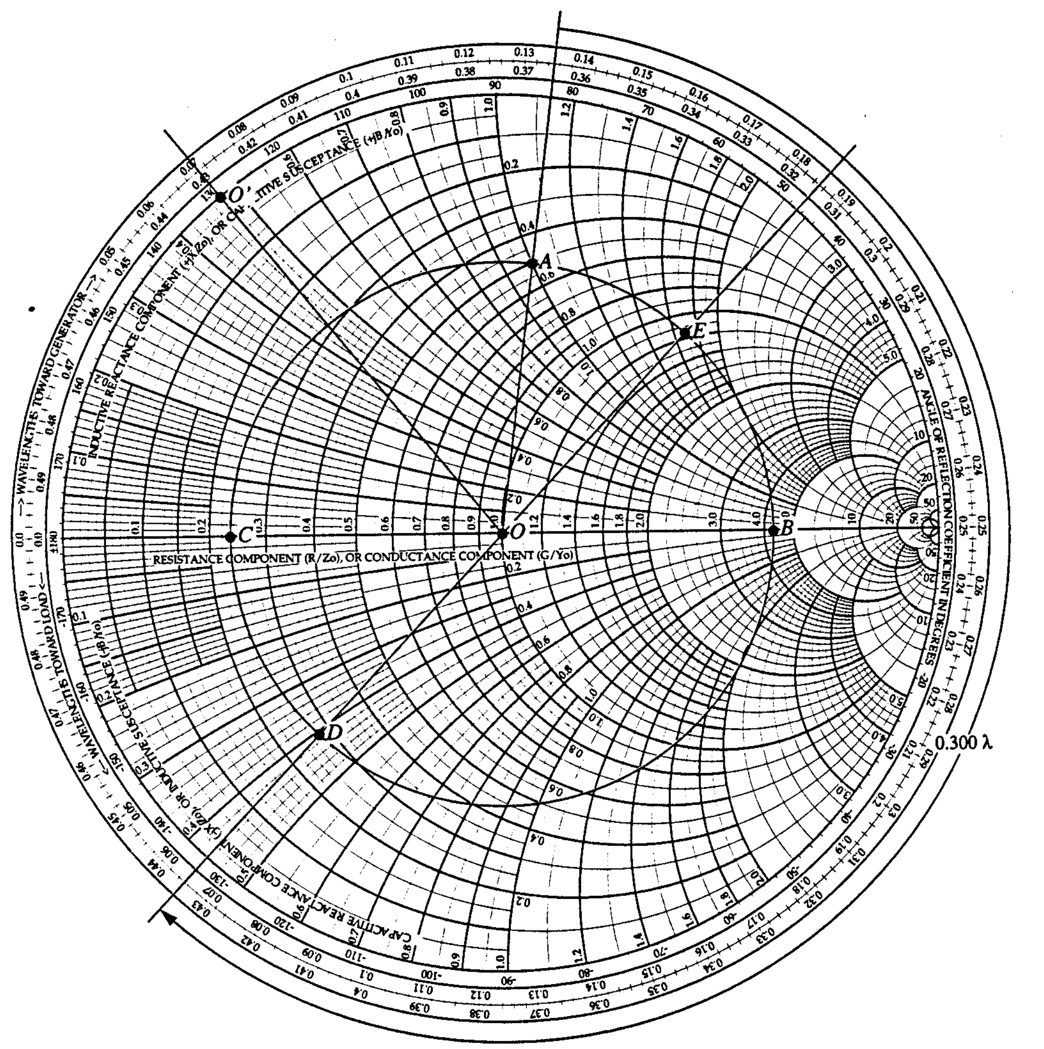


Figure 2-26: Solution for Example 2-10. Point A represents a normalized load $z_L = 0.5 + j1$ at 0.135λ on the WTG scale. At A, $\theta_r = 83°$ and $|\Gamma| = d_A/d_{O'} = \overline{OA}/\overline{OO'} = 0.62$. At B, the standing-wave ratio is $S = 4.26$. The distance from A to B gives $l_{max} = 0.115\lambda$ and from A to C gives $l_{min} = 0.365\lambda$. Point D represents the normalized input impedance $z_{in} = 0.28 - j0.40$, and point E represents the normalized input admittance $y_{in} = 1.15 + j1.7$.

of the line between points O and A and the length $d_{O'}$ of the line between points O and O', where O' is an arbitrary point on the $r_L = 0$ circle. The length $d_{O'}$ is equal to the radius of the $|\Gamma| = 1$ circle. The magnitude of Γ is then obtained from $|\Gamma| = d_A/d_{O'} = 0.62$. Hence,

$$\Gamma = 0.62e^{j83°} = 0.62\angle 83°. \tag{2.121}$$

(b) Using a compass, the SWR circle with center at

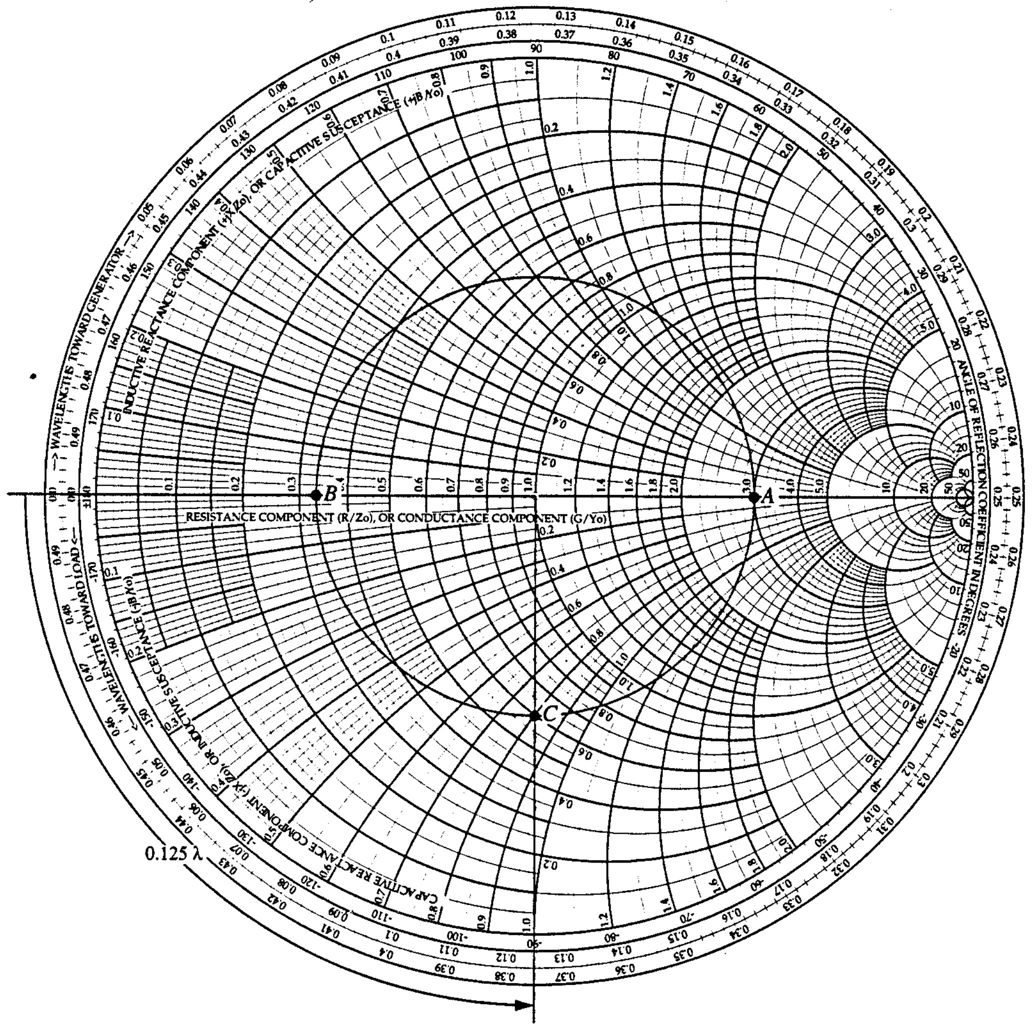


Figure 2-27: Solution for Example 2-11. Point A denotes that $S = 3$, point B represents the location of the voltage minimum, and point C represents the load at 0.125λ on the WTL scale from point B. At C, $z_L = 0.6 - j0.8$.

Q2.21 What line length corresponds to one complete rotation around the Smith chart? Why?

Q2.22 What points on the SWR circle correspond to the locations of the voltage maxima and minima on the line and why?

Q2.23 Given a normalized impedance z_L, how do you use the Smith chart to find the corresponding normalized admittance $y_L = 1/z_L$?