PLANE WAVES

phasors - 1 freq. only
time-indepent

Waves propagating in the free space / uniform material. (uniform properties at all points of a plane ⊥ prop.)

If the time variation is sinusoidal with ω, $\bar{E}, \bar{D}, \bar{B}, \bar{H}, \bar{J}$ can be represented by a time-independent phasor ($f(x,y,z)$)

e.g. $\bar{E}(x,y,z,t) = \mathrm{Re}\,[\tilde{E}(x,y,z)\,e^{j\omega t}]$

$$\Downarrow$$

$$\frac{\partial}{\partial t} \longrightarrow j\omega$$

Phasor form of Maxwell's laws.

$\nabla \cdot \tilde{E} = \tilde{\rho}_v / \varepsilon$, $\nabla \times \tilde{E} = -j\omega\mu \tilde{H}$, $\nabla \cdot \tilde{H} = 0$, $\nabla \times \tilde{H} = \tilde{J} + j\omega\varepsilon \tilde{E}$

$(\tilde{D} = \varepsilon \tilde{E},\ \tilde{B} = \mu \tilde{H})$

$$\nabla \times \tilde{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}(\ldots) + \hat{y}(\ldots) + \hat{z}(\ldots)$$
$$= -j\omega\mu(\hat{x}H_x + \hat{y}H_y + \hat{z}H_z)$$

Triangular Rule for External Product

$$\begin{tikzcd} & \hat{x} \arrow[dr] & \\ \hat{z} \arrow[ur] & & \hat{y} \arrow[ll] \end{tikzcd}$$

Multiplying two terms clockwise gives the third term (e.g. $\hat{x} \times \hat{y} = \hat{z}$)

Multiplying two terms counterclockwise gives the opposite of the third term (e.g. $\hat{y} \times \hat{x} = -\hat{z}$)

$$\hat{x}\left(\frac{dE_x}{dy} - \frac{dE_y}{dx}\right) - \hat{y}(\ldots) + \hat{z}(\ldots)$$

Complex Permittivity for Media with finite conductivity σ :

$$\tilde{J} = \sigma \tilde{E} \quad , \quad \nabla \times \tilde{H} = \tilde{J} + j\omega\varepsilon \tilde{E} = (\sigma + j\omega\varepsilon)\tilde{E} =$$

$$= j\omega \left(\varepsilon - j \frac{\sigma}{\omega} \right) \tilde{E} = j\omega \varepsilon_c \tilde{E}$$

Define: Complex Permittivity: $\varepsilon_c \triangleq \varepsilon - j \frac{\sigma}{\omega} =$

$$= \underset{Re(\varepsilon_c)}{\varepsilon'} - j \underset{Im(\varepsilon_c)}{\varepsilon''}$$

Lossless Medium: $\sigma = 0 \Rightarrow \varepsilon'' = 0 \Rightarrow \varepsilon_c = \varepsilon' = \varepsilon = \varepsilon_r \varepsilon_0$

Since, a uniform plane wave has uniform properties at all points across an infinite plane, assuming a propagation to $\hat{z} \Rightarrow \frac{\partial \tilde{E}_{x,y,z}}{\partial x} = \frac{\partial \tilde{E}_{x,y,z}}{\partial y} = 0$

It can be proven that field components $\perp$ uniform plane are zero (TEM)

i.e. $\tilde{E} = \tilde{E}_x^+ \hat{x} = E_{x0}^+ e^{-jkz}$ ~~[illegible]~~ $(\longrightarrow +\hat{z})$

$\nabla \times \tilde{E} = -j\omega\mu \tilde{H} \Rightarrow$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \tilde{E}_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu \left(\hat{x}\tilde{H}_x + \hat{y}\tilde{H}_y + \hat{z} H_z \right)$$

Since plane wave propagates to $+\hat{z} \rightarrow \frac{\partial \tilde{E}_x^+(z)}{\partial x} = \frac{\partial \tilde{E}_x^+}{\partial y} =$

$$\Rightarrow \tilde{H}_x = 0, \quad \tilde{H}_y = \frac{k}{\omega\mu} E_{x0}^+ e^{-jkz} = H_{y0}^+ e^{-jkz}$$

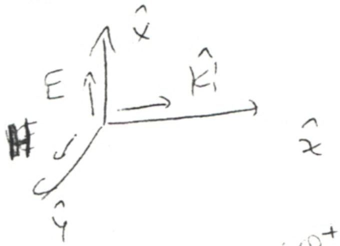
// V_0^+, I_0^+ related to char. impedance Z_0 (3)

<u>Define:</u> intrinsic impedance of a lossless medium

$$\eta \triangleq \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} \underset{\sigma=0}{=} \sqrt{\frac{\mu}{\varepsilon}}\,(\Omega) = \frac{\overset{\overset{377}{\downarrow}}{120\pi}}{\sqrt{\varepsilon_r}}\sqrt{\mu_r}$$

$$\frac{E_{x0}^+}{H_{y0}^+} = \underset{\substack{\downarrow \\ \text{intrinsic} \\ \text{impedance}}}{\eta} \rightarrow \left\{ \begin{array}{l} \tilde{E}(z) = \hat{x}\,\tilde{E}_x^+(z) = \hat{x}\,E_{x0}^+ e^{-jkz} \\ \tilde{H}(z) = \hat{y}\,\dfrac{\tilde{E}_x^+(z)}{\eta} = \hat{y}\,\dfrac{E_{x0}^+}{\eta} e^{-jkz} \end{array} \right\}$$

Perpendicular to each other and to the propagation direction

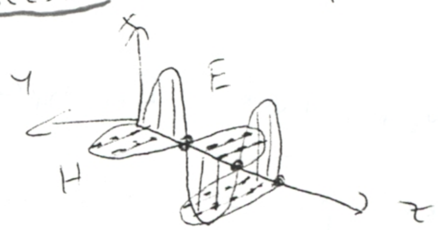


If E_{x0}^+ complex $= |E_{x0}^+| e^{j\varphi^+}$

$$\bar{E}(z,t) = \mathrm{Re}\{\tilde{E}(z) e^{j\omega t}\} = \hat{x}\,|E_{x0}^+| \cos(\omega t - kz + \varphi^+)$$

$$\bar{H}(z,t) = \mathrm{Re}\{\tilde{H}(z) e^{j\omega t}\} = \hat{y}\,\frac{|E_{x0}^+|}{\eta} \cos(\omega t - kz + \varphi^+)$$

Same dependence on z, t $\Rightarrow$ in phase $\Rightarrow$ max at the same position $\Rightarrow$ In phase property is a characteristic of <u>lossless</u> media propagation



evolution in time

Phase velocity: $u_p \triangleq \frac{\omega}{\ldots} = \frac{\omega}{\ldots} = \frac{1}{\ldots} = c_0 = 3\times10^8$ m/sec wavelength (distance between two successive min/max) $\Rightarrow \lambda = \frac{2\pi}{k} = \frac{u_p}{f}$ (m)

Vacuum: $\varepsilon = \varepsilon_0,\ \mu = \mu_0 \Rightarrow \eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega \simeq 120\pi$

Intrinsic impedance of free space

For a general propagation direction $\hat{k}$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} \quad \text{(or)} \quad \tilde{E} = -\eta\,\hat{k} \times \tilde{H}$$

VALID FOR LOSSY AND LOSSLESS MEDIA!!

e.g. $\hat{k} = \hat{z} \Rightarrow \tilde{E} = \hat{x}\tilde{E}_x^+(z) \Rightarrow$

$$\Rightarrow \tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \frac{1}{\eta}(\hat{z} \times \hat{x})\tilde{E}_x^+(z) = \hat{y}\frac{\tilde{E}_x^+(z)}{\eta}$$

$\tilde{E} = \hat{x}\tilde{E}_x^-(z)$ $\quad \underline{\hat{k} = -\hat{z}}$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \frac{1}{\eta}(-\hat{z} \times \hat{x})\tilde{E}_x^-(z) = -\hat{y}\frac{\tilde{E}_x^-(z)}{\eta}$$

$\tilde{E} = \hat{x}\tilde{E}_x^+(z) + \hat{y}\tilde{E}_y^+(z) \Rightarrow$ $\qquad \tilde{E}_x^+ = E_x^0 e^{-jkz}$

$$\Rightarrow \tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \frac{1}{\eta}\hat{z} \times (\hat{x}\tilde{E}_x^+(z) + \hat{y}\tilde{E}_y^+(z)) =$$

$$= \hat{y}\frac{\tilde{E}_x^+(z)}{\eta} - \hat{x}\frac{\tilde{E}_y^+(z)}{\eta}$$

Orthogonality is still maintained

Numerical Example for TV Broadcast ($f = 300$ MHz)

The E-field phasor of a uniform plane wave travelling in a lossless medium with $\varepsilon_r = 4$ is given by: $\tilde{E} = \hat{z}\,10 e^{-j4\pi y}$ (mV/m). Determine $\tilde{H}(z)$ and the instantaneous expression for $\bar{E}(y,t)$.

Solution

$$\tilde{E} = \hat{z}\ 10\ e^{-j4\pi y}$$

↳ prop. to $+\hat{y}$ direction $\Rightarrow \hat{k}_i = \hat{y}$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{120\pi}{2} = 188.5\ \Omega$$

$$\tilde{H} = \frac{1}{\eta}\hat{k}_i \times \tilde{E} = \frac{1}{188.5}\left(\hat{y} \times \hat{z}\, 10 e^{-j4\pi y}\right) =$$

$$= 0.053\ e^{-j4\pi y}\ \hat{x} \quad (mA/m)$$

$$\bar{E}(y,t) = Re\left(\tilde{E}(y)\, e^{j\omega t}\right) =$$

$$= Re\left[\hat{z}\, 10 e^{-j4\pi y}\ e^{j2\pi 300\times 10^6 t}\right]$$

$$= 10 \cos\left(6\times 10^8 \pi t - 4\pi y\right)\ \hat{z} \quad (mV/m)$$

Wavelength: $\lambda = \frac{u_p}{f} = \frac{1.5\times 10^8}{300\times 10^6} = 0.5\ m$

Phase velocity: $u_p = \frac{c_0}{\sqrt{\varepsilon_r}} = 3\times 10^8/2 = 1.5\times 10^8\ m/sec$

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MEMORANDUM

Date:

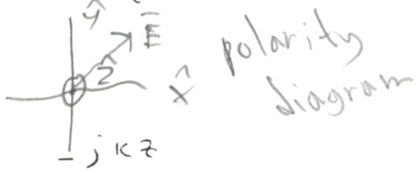
To:

From:

Subject:

WAVE POLARIZATION

Polarization of a uniform wave describes the shape and locus of the tip of $\bar{E}$ vector (in the plane orthogonal to the direction of propagation) at a given point in space as a function of time.



e.g. $\tilde{E}(z) = \hat{x}\,\tilde{E}_x(z) + \hat{y}\,\tilde{E}_y(z)$

$$\tilde{E}_x(z) = E_{xo}\, e^{-jkz}, \quad \tilde{E}_y(z) = E_{yo}\, e^{-jkz}$$

E_{xo}, E_{yo} are complex amplitudes of $\tilde{E}_x(z)$, $\tilde{E}_y(z)$

Assume phase difference δ.

$$E_{xo} = a_x, \quad E_{yo} = a_y e^{j\delta} \quad ; \quad a_x = |E_{xo}|, \; a_y = |E_{yo}|$$

$$\Downarrow$$

$$\tilde{E}(z) = (\hat{x} a_x + \hat{y} a_y e^{j\delta})\, e^{-jkz}$$

$$\Downarrow$$

Instantaneous field:

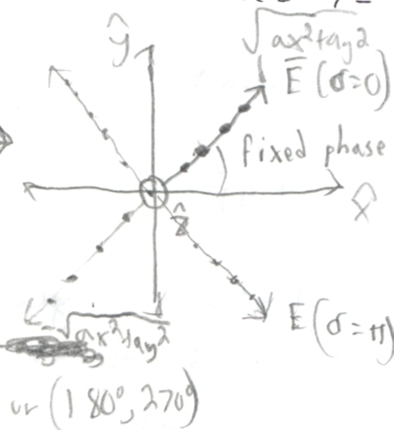
$$\longrightarrow \quad \bar{E}(z,t) = \mathrm{Re}\left[\tilde{E}(z) e^{j\omega t}\right] = \hat{x}\, a_x \cos(\omega t - kz) + \hat{y}\, a_y \cos(\omega t - kz + \delta)$$

Modulus: $|\bar{E}(z,t)| = \sqrt{E_x^2(z,t) + E_y^2(z,t)} = \left[a_x^2 \cos^2(\omega t - kz) + a_y^2 \cos^2(\omega t - kz + \delta)\right]^{1/2}$

Phase: $\Psi(z,t) = \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$

phase is between vertical comp. and horizontal comp.

A wave is <u>linearly polarized</u> if $E_x(z,t)$, $E_y(z,t)$ are in phase ($\delta=0$) or out of phase ($\delta=\pi$) and $a_x = a_y = \ldots$



$$\bar{E}(0,t) = (\hat{x}a_x + \hat{y}a_y)\cos(\omega t) \quad \text{In-phase}$$

$$\bar{E}(0,t) = (\hat{x}a_x - \hat{y}a_y)\cos(\omega t) \quad \text{Out-of-phase}$$

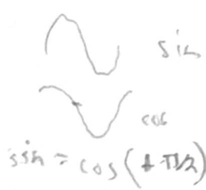
$$|\bar{E}(0,t)| = \sqrt{a_x^2+a_y^2}\cos\omega t \rightarrow \left(-\sqrt{a_x^2+a_y^2}, \sqrt{a_x^2+a_y^2}\right)$$

$$\psi = \tan^{-1}\left(\frac{\pm a_y}{a_x}\right) \text{ not } f(t) \quad \left(\begin{matrix} + & \text{in} \\ - & \text{out} \end{matrix}\right) \rightarrow \begin{matrix} (0^\circ, 90^\circ) \text{ or } (180^\circ, 270^\circ) \\ (90, 180) \text{ or } (270, 360) \end{matrix}$$

Circular

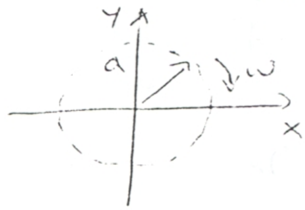
$a_x = a_y$

→ Left-hand circular ($\delta = \pi/2$)



$$\tilde{E}(z) = (\hat{x}a + \hat{y}a e^{j\pi/2})e^{-jk} = (a\hat{x} + a\hat{y}j)e^{-jkz}$$

$$\bar{E}(z,t) = \hat{x}a\cos(\omega t - kz) - \hat{y}a\sin(\omega t - kz)$$



$$|\bar{E}(z,t)| = a \quad \psi(z,t) = -(\omega t - kz)$$

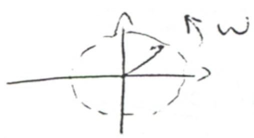
$(\cos^2+\sin^2=1)$ not $f(t)$ $\quad \tan^{-1}\left(\frac{-\sin(\omega t-kz)}{\cos(\omega t-kz)}\right)$

(Inclination decreases with increasing time)

Right-hand circular ($\delta = -\pi/2$)

$a_x = a_y$, $\delta = -\pi/2$

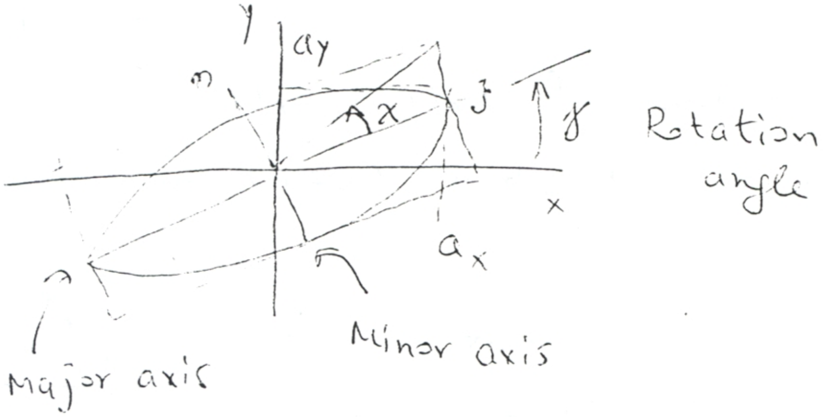
$|\bar{E}(z,t)| = a$, $\psi(z,t) = \omega t - kz$



Elliptical polarization

General case, $a_x \neq 0$, $a_y \neq 0$ and $\delta \neq 0$

The shape of the ellipse and its handedness (left-hand or right-hand) are determined by values of a_y/a_x and the polarization phase difference δ



Major axis a_ξ along ξ-direction.

Minor axis a_η along η-direction

Rotation angle γ is defined as the angle between the major axis of the ellipse and a reference direction (chosen here to be the x-axis) $\left(-\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}\right)$

The shape and the handedness are characterized by the ellipticity angle χ:

$$\tan\chi = \pm\frac{a_\eta}{a_\xi} = \pm\frac{1}{R} \qquad \left(\begin{array}{l} + \rightarrow \text{LH rotation} \\ - \rightarrow \text{RH rotation} \end{array}\right.$$

$R = \frac{a_\xi}{a_\eta}$ ~ axial ratio $(\in [1, \infty))$

Define: $\tan\psi_0 = \frac{a_y}{a_x} \qquad \left(0 \le \psi_0 \le \frac{\pi}{2}\right)$

$\tan 2\gamma = (\tan 2\psi_0)\cos\delta$, $\sin 2\chi = (\sin 2\psi_0)\sin\delta$

Positive values of χ ($\sin\delta > 0$) $\rightarrow$ LH

Negative — — $\rightarrow$ RH

$$\left(\begin{array}{l} -\frac{\pi}{2} \le \gamma \le \frac{\pi}{2} \\ -\frac{\pi}{4} \le \chi \le \frac{\pi}{4} \end{array}\right)$$

Also:

$$\boxed{\begin{array}{ll} \gamma > 0 & \text{if } \cos\delta > 0 \\ \gamma < 0 & \text{if } \cos\delta < 0 \end{array}}$$

way of eliminating one of two solutions.

eg. Find polarization state of a plane wave:

$$\bar{E}(z,t) = \hat{x}\, 3\cos(\omega t - kz + 30^\circ) - \hat{y}\, 4\sin(\omega t - kz + 45^\circ) \text{ mV/m}$$

STEP 1 change to a cosine reference

elliptical polar. not on tests

$$\bar{E} = \hat{x}\, 3\cos(\omega t - kz + 30^\circ) - \hat{y}\, 4\cos(\omega t - kz + 45^\circ - 90^\circ)$$

$$= \hat{x}\, 3\cos(\omega t - kz + 30^\circ) - \hat{y}\, 4\cos(\omega t - kz - 45^\circ)$$

STEP 2 Find corresponding phasor

$$\tilde{E}(z) = \hat{x}\, 3e^{-jkz} e^{j30^\circ} - \hat{y}\, 4e^{-jkz} e^{-j45^\circ}$$

$$= \hat{x}\, 3e^{-jkz} e^{j30^\circ} + \hat{y}\, 4e^{-jkz} e^{-j45^\circ} e^{j180^\circ}$$

$$= \hat{x}\, 3e^{-jkz} e^{j30^\circ} + \hat{y}\, 4e^{-jkz} e^{j135^\circ}$$

STEP 3 Phase angles $\delta_x = 30^\circ$; $\delta_y = 135^\circ$

$\hookrightarrow$ phase difference $\delta = \delta_y - \delta_x = 105^\circ$

Auxiliary $\psi_0 = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$

STEP 4 $\tan 2\gamma = (\tan 2\psi_0)\cos\delta = \tan 106.2^\circ \cos 105^\circ = 0.89$

$(-90^\circ \le \gamma \le 90^\circ)$

$\Downarrow$

$\gamma = 20.6^\circ$ or $\gamma = -69.2^\circ$

Since $\cos\delta < 0 \rightarrow \gamma < 0 \rightarrow \gamma = -69.2^\circ$

$\sin 2\chi = (\sin 2\psi_0)\sin\delta = \sin 106.2^\circ \sin 105^\circ = 0.93$

$(-45^\circ \le \chi \le 45^\circ)$

$\Downarrow$

$\chi = 34^\circ$

!

elliptically polarized

and (+) polarity $\rightarrow$ LHS

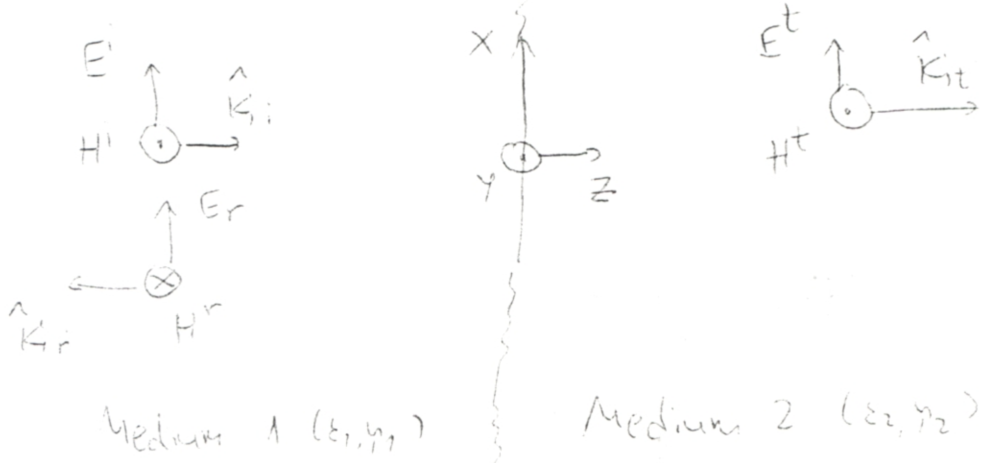
$\chi = 45^\circ \rightarrow$ LHC

$\chi = 0^\circ \rightarrow$ Linear

$\chi = -45^\circ \rightarrow$ RHC

MEDIUM - INTERFACES

a) Normal Incidence



$$\tilde{E}_1(z) = \tilde{E}^i(z) + \tilde{E}^r(z)$$

$$= \hat{x}\,(E_0^i e^{-jk_1 z} + E_0^r e^{jk_1 z}) \qquad k_1 = w\sqrt{\mu_1 \varepsilon_1}$$

$$\tilde{E}_2(z) = \tilde{E}^t(z) = \hat{x}\, E_0^t e^{-jk_2 z}$$

$$\tilde{H}_2(z) = \tilde{H}^t(z) = \hat{y}\,\frac{E_0^t}{\eta_2} e^{-jk_2 z}$$

$$k_2 = w\sqrt{\mu_2 \varepsilon_2}$$

$$\tilde{H}_1(z) = \tilde{H}^i(z) + \tilde{H}^r(z) =$$

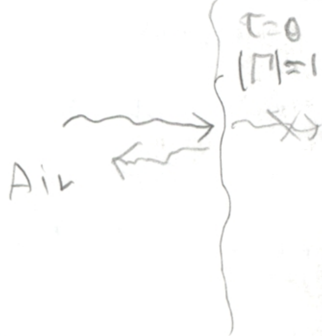
$$= \hat{y}\,\frac{1}{\eta_1}(E_0^i e^{-jk_1 z} - E_0^r e^{jk_1 z})$$

At the boundary $(z=0)$, tangential components of $\tilde{E}, \tilde{H}$ are continuous

$$\Gamma = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \qquad \tau = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$\downarrow$ Refl. Coefficient

$\downarrow$ Trans. Coefficient



Metal/plastic package

PEC (SC) $\rightarrow |\Gamma| =$

or $L, C \rightarrow |\Gamma| =$

(high - ε_r plastic

(Lossless media $\rightsquigarrow \eta_1, \eta_2 \in \mathbb{R} \rightsquigarrow \Gamma, \tau \in \mathbb{R}$)

$\tau = 1 + \Gamma$

Conventional conduct

thickness ~ 5-$10\ \delta_s$

For nonmagnetic media $(\mu_{r_1} = \mu_{r_2} = 1)$

air/Si { lossy conductor { 5-10 δ_s

Standing-wave ratio. $S = \dfrac{1+|\Gamma|}{1-|\Gamma|}$ ②

Power Flow in Lossless

$$\tilde{S}^{i}_{av} = \hat{z}\frac{|E^i_0|^2}{2\eta_1}, \quad \tilde{S}^{r}_{av} = -|\Gamma|^2 \tilde{S}^{i}_{av}\hat{z}, \quad \tilde{S}^{t}_{av} = \hat{z}\,|\tau|^2 \frac{|E^i_0|^2}{2\eta_2}$$

$$\tilde{S}^{i}_{av} + \tilde{S}^{r}_{av} \equiv \tilde{S}^{t}_{av} \quad \text{(No Loss)}$$

For Lossy Medium:

(#1) $\tilde{E}_1(z) = \hat{x}\, E^i_0 (e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z})$

$\tilde{H}_1(z) = \hat{y}\, \dfrac{E^i_0}{\eta_{c_1}} (e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z})$

where: $\begin{cases} \gamma_1 = \alpha_1 + j\beta_1 \\ \gamma_2 = \alpha_2 + j\beta_2 \end{cases}$

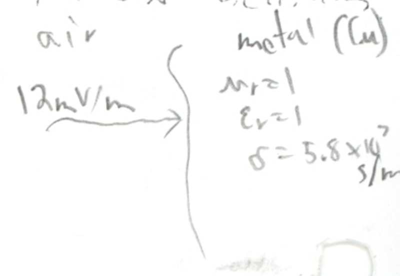
(#2) $\tilde{E}_2(z) = \hat{x}\,\tau\, E^i_0\, e^{-\gamma_2 z}$

$\tilde{H}_2(z) = \hat{y}\,\tau\, \dfrac{E^i_0}{\eta_{c_2}}\, e^{-\gamma_2 z}$

$$\Gamma = \frac{\eta_{c_2} - \eta_{c_1}}{\eta_{c_2} + \eta_{c_1}}, \quad \tau = 1 + \Gamma = \frac{2\eta_{c_2}}{\eta_{c_1} + \eta_{c_2}} \quad (\eta_{c_1}, \eta_{c_2} \in \mathbb{C} \Rightarrow \Gamma, \tau \in \mathbb{C})$$

Example

A 1 GHz $\hat{x}$-polarized TEM wave $\rightarrow +\hat{z}$ is incident in air upon a metal surface coincident with x-y plane at $z=0$. The amplitude of the el. field of the incident wave is 12 mV/m. The metal surface is copper with $\mu_r = 1$, $\varepsilon_r = 1$, $\sigma = 5.8\times10^7$ S/m. Find expressions for instantaneous $\bar{E}$, $\bar{H}$ fields in air medium.



Solution

#1 ⇒ Air $\quad \alpha = 0, \ \beta = k_1 = \dfrac{\omega}{c} = \dfrac{2\pi\times10^9}{3\times10^8} = \dfrac{20\pi}{3}$ rad/m

$= \omega\sqrt{\mu_0\varepsilon_0}$

$\eta_1 = \eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\,\Omega, \quad \lambda = \dfrac{2\pi}{k_1} = 0.3\,m$

Medium 2

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{5.8\times10^7}{2\pi\times10^9\times\frac{10^{-9}}{36\pi}} = 10^9 \gg 1 \rightarrow \text{excellent conductor}$$

$$\eta_{c_2} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\left[\frac{\pi\times10^9\times4\pi\times10^{-7}}{5.8\times10^7}\right]^{1/2} =$$

$$= 8.25\,(1+j)\ (\text{m}\Omega)$$

$$\Gamma = \frac{\eta_{c2}-\eta_1}{\eta_{c2}+\eta_1} \simeq -1 \quad \text{(expected for excellent conductor)}$$

$$\tilde{E}_1(z) = \hat{x}\,E_0^i\,(e^{-jk_1z}-e^{jk_1z}) = -\hat{x}\,j\,2E_0^i\sin k_1z$$

$$\tilde{H}_1(z) = \hat{y}\,\frac{E_0^i}{\eta_1}\,(e^{-jk_1z}+e^{jk_1z}) = \hat{y}\,2\,\frac{E_0^i}{\eta_1}\cos k_1z$$

$$E_0^i = 12 \text{ mV/m}$$

Instantaneous fields

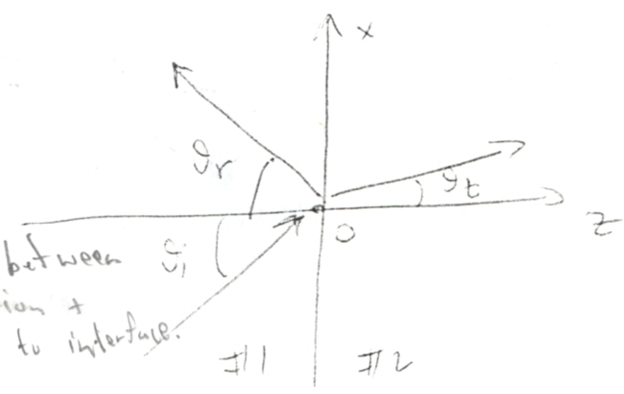
$$\bar{E}_1(z,t) = \text{Re}[\tilde{E}_1(z)e^{j\omega t}] = \hat{x}\,2E_0^i\,\underline{\sin}\,k_1z\,\underline{\sin}\,\omega t =$$

$$= \hat{x}\,24\sin(20\pi z/3)\sin(2\pi\times10^9 t) \quad (\text{mV/m})$$

$$\bar{H}_1(z,t) = \text{Re}[\tilde{H}_1(z)e^{j\omega t}] = \hat{y}\,2\,\frac{E_0^i}{\eta_1}\cos k_1z\cos\omega t =$$

$$= \hat{y}\,64\cos(20\pi z/3)\cos(2\pi\times10^9 t) \quad (\mu\text{A/m})$$

SNELL'S LAWS



angle between propagation + normal to interface.

$\theta_i = \theta_r$ (Law of reflection)

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \underbrace{\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}}_{\text{lossless}}$$

(Law of refraction)

Define index of refraction (optical)

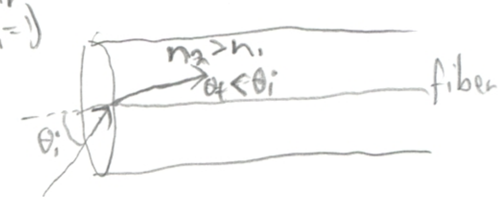
$$n = \frac{c}{u_p} = \underbrace{\sqrt{\mu_r \varepsilon_r}}_{\text{lossless}} \Rightarrow \frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\mu_{r_1}\varepsilon_{r_1}}{\mu_{r_2}\varepsilon_{r_2}}}$$

A material is often refered to as a more dense than a second material if $n_1 > n_2$.

At normal incidence $(\theta_i = 0) \Rightarrow \theta_t = 0$

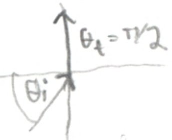
At oblique incidence : $\theta_t < \theta_i$ when $n_2 > n_1$

and $\theta_t > \theta_i$ when $n_2 < n_1$



The value of incidence θ_i corresponding to $\theta_t = \pi/2$ is critical angle θ_c

$$\sin\theta_c = \frac{n_2}{n_1} \left(= \sqrt{\frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}} \quad (\text{for } \mu_1 = \mu_2) \right)$$



If $\theta_i > \theta_c$ → total reflection and the refracted wave becomes a surface wave.

$\theta_i = \theta_c \rightarrow \theta_t = \frac{\pi}{2}$ (prop. surface wave)

$\theta_i > \theta_c \rightarrow \underbrace{\theta_t = \frac{\pi}{2}}_{\text{complex angle}}$ (atten. surface wave) energy conc. at incident contact destroys lin

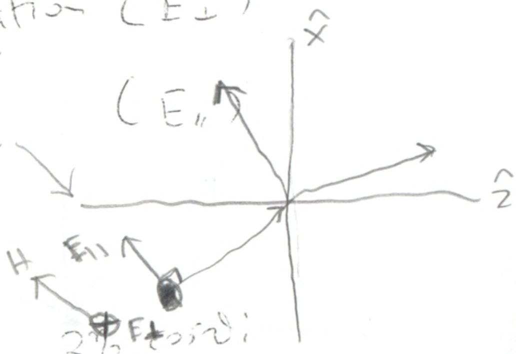
→ Wave Reflection and Transmission at oblique Incidence

The plane of incidence is defined as the plane containing the normal to the boundary and the direction of propagation of the incident wave.

If $\bar{E} \perp$ plane of incidence ⇒ TE polarization $(E_\perp)$
Transverse electric

If $\bar{E} \parallel$ plane of incidence ⇒ TM
Transverse Magnetic

(A) Perpendicular polarization



$\theta_r = \theta_i$, $\sin\theta_t / \sin\theta_i = \frac{n_1}{n_2}$

$$\Gamma_\perp = \frac{E^r_{\perp 0}}{E^i_{\perp 0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \neq 0 \quad \tau_\perp = \frac{E^t_{\perp 0}}{E^i_{\perp 0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$\tau_\perp = 1 + \Gamma_\perp$

(B) Parallel Polarization

$$\Gamma_{\parallel} = \frac{E^r_{\parallel 0}}{E^i_{\parallel 0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_{\parallel} = \frac{E^t_{\parallel 0}}{E^i_{\parallel 0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\left(\tau_{\parallel} = (1+\Gamma_{\parallel}) \frac{\cos\theta_i}{\cos\theta_t} \right)$$

For nonmagnetic: $$\Gamma_{\parallel} = \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}{(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}$$

$\Gamma_{\parallel} = 0$ at an angle θ_B for θ_i (Brewster angle)

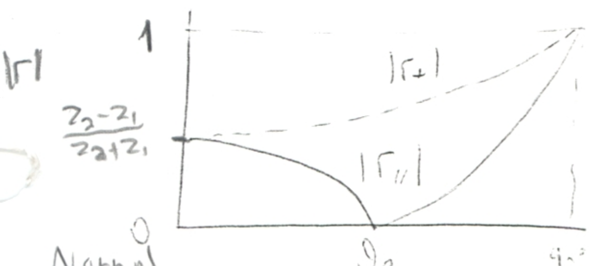
↳ total transmission to medium (2)

$$\sin\theta_{B_\parallel} = \sqrt{\frac{1 - (\varepsilon_1\mu_2/\varepsilon_2\mu_1)}{1 - (\varepsilon_1/\varepsilon_2)^2}}$$

For nonmagnetic:

$$\theta_{B_\parallel} = \sin^{-1}\sqrt{\frac{1}{1+(\varepsilon_1/\varepsilon_2)}} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$\nexists\ \theta_{B_\perp}$ for nonmagnetic materials!!!

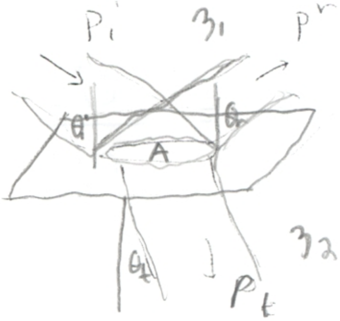


$|\Gamma_\parallel| = |\Gamma_\perp|$ at $\theta_i = 90°$ (grazing incidence)

$|\Gamma_\parallel| = |\Gamma_\perp|$ for $\theta_i = 0°$ (normal

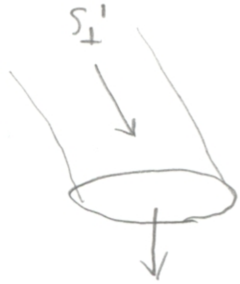
Reflectivity and Transmissivity

$$\underset{(W/m^2)}{S_\perp^i} = \frac{|E_{\perp 0}^i|^2}{2\eta_1}, \quad S_\perp^r = \frac{|E_{\perp 0}^r|^2}{2\eta_1}, \quad S_\perp^t = \frac{|E_{\perp 0}^t|^2}{2\eta_2}$$



Cross-sectional area of beams

$$A_i = A\cos\vartheta_i, \quad A_r = A\cos\vartheta_r, \quad A_t = A\cos\vartheta_t$$



Average powers

$$P_\perp^i = S_\perp^i A_i = \frac{|E_{\perp 0}^i|^2}{2\eta_1} \underline{A\cos\vartheta_i}$$

$$P_\perp^r = S_\perp^r A_r = \frac{|E_{\perp 0}^r|^2}{2\eta_1} \underline{A\cos\vartheta_r}$$

$$P_\perp^t = S_\perp^t A_t = \frac{|E_{\perp 0}^t|^2}{2\eta_2} \underline{A\cos\vartheta_t}$$

Reflectivity:

$$R_\perp = \frac{P_\perp^r}{P_\perp^i} = |\Gamma_\perp|^2 \qquad \left(// \quad R_\parallel = \frac{P_\parallel^r}{P_\parallel^i} = |\Gamma_\parallel|^2 \right)$$

Transmissivity:

$$T_\perp = \frac{P_\perp^t}{P_\perp^i} = |\tau_\perp|^2 \left(\frac{\eta_1 \cos\vartheta_t}{\eta_2 \cos\vartheta_i} \right)$$

$$T_\parallel = \frac{P_\parallel^t}{P_\parallel^i} = |\tau_\parallel|^2 \left(\frac{\eta_1 \cos\vartheta_i}{\eta_2 \cos\vartheta_t} \right)$$

Conservation of power:

$$P_\perp^i = P_\perp^r + P_\perp^t \quad \Rightarrow \quad R_\perp + T_\perp = 1$$

$$P_\parallel^i = P_\parallel^r + P_\parallel^t \quad \Rightarrow \quad R_\parallel + T_\parallel = 1$$

$\Gamma = 0$

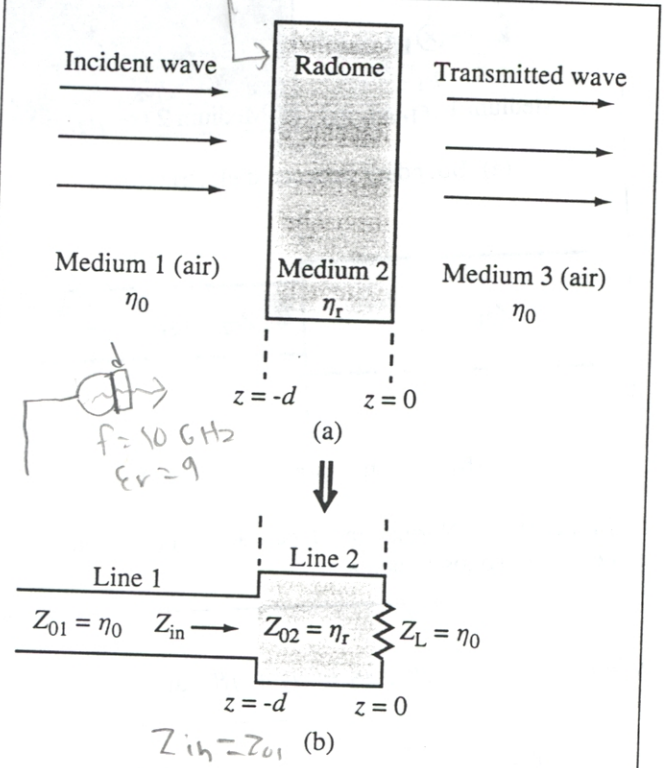


Figure 8-6: (a) Planar section of the radome of Fig. 8-5 at an expanded scale and (b) its transmission-line equivalent model (Example 8-1).

Requiring the radome to "appear" transparent to the incident wave simply means that the reflection coefficient must be zero at $z = -d$, thereby achieving total transmission of the incident power into medium 3. Since $Z_L = \eta_0$ in Fig. 8-6(b), no reflection will take place at $z = -d$ if $Z_{in} = \eta_0$, which can be realized by choosing $d = n\lambda_2/2$ [see Section 2-7.4], where λ_2 is the wavelength in medium 2 and n is a positive integer. At 10 GHz, the wavelength in air is $\lambda_0 = c/f = 3$ cm, and in the radome material

$$\lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{3 \text{ cm}}{3} = 1 \text{ cm}.$$

$d = m.5$ cm

iii. $d \ll \lambda$

$\Gamma = \frac{jk_2 d}{2}\left(\frac{Z_2}{Z_1} - \frac{Z_1}{Z_2}\right)$

$= jCd$

$Z_2 \quad Z_3$

$Z_{in}(z=d) = Z_2\left(\frac{Z_3 + jZ_2 \tan\beta_2 l}{Z_2 + jZ_3 \tan\beta_2 l}\right)$

i. $d = m\frac{\lambda_2}{2}$

$\beta_2 l = \frac{2\pi}{\lambda_2} \cdot \frac{m\lambda_2}{2} = m\pi$

Hence, if we choose $d = 5\lambda_2/2 = 2.5$ cm, we will satisfy both the no-reflection and the mechanical integrity requirements. ■

Example 8-2 Yellow Light Incident upon a Glass Surface

A beam of yellow light with wavelength of 0.6 μm is normally incident in air upon a glass surface. If the surface is situated in the plane $z = 0$ and the relative permittivity of glass is 2.25, determine:

(a) the locations of the electric field maxima in medium 1 (air),
(b) the standing-wave ratio, and
(c) the fraction of the incident power transmitted into the glass medium.

Solution: (a) We begin by determining the value of η_1, η_2, and Γ:

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 120\pi \ (\Omega),$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{\sqrt{2.25}} = 80\pi \ (\Omega),$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{80\pi - 120\pi}{80\pi + 120\pi} = -0.2.$$

Hence, $|\Gamma| = 0.2$ and $\theta_r = \pi$. From Eq. (8.16), the electric-field magnitude is a maximum at

$$l_{max} = \frac{\theta_r \lambda_1}{4\pi} + n\frac{\lambda_1}{2} = \frac{\lambda_1}{4} + n\frac{\lambda_1}{2} \qquad (n = 0, 1, 2, \ldots)$$

with $\lambda_1 = 0.6$ μm.

(b)

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.2}{1 - 0.2} = 1.5.$$

(c) The fraction of the incident power transmitted into the glass medium is equal to the ratio of the transmitted power density, given by Eq. (8.20), to the incident power density, $S^{i}_{av} = |E^{i}_{0}|^2/2\eta_1$:

$$\frac{S_{av_2}}{S^{i}_{av}} = \tau^2 \frac{|E^{i}_{0}|^2}{2\eta_2} \Bigg/ \left[\frac{|E^{i}_{0}|^2}{2\eta_1}\right] = \tau^2 \frac{\eta_1}{\eta_2}.$$

In view of Eq. (8.21),

$$\frac{S_{av_2}}{S^{i}_{av}} = 1 - |\Gamma|^2 = 1 - (0.2)^2 = 0.96, \text{ or } 96\%. \blacksquare$$

8-1.4 Boundary between Lossy Media

In Section 8-1.1 we considered a plane wave in a lossless medium incident normally on a planar boundary of another lossless medium. We will now generalize our expressions to lossy media. In a medium with constitutive parameters $(\varepsilon, \mu, \sigma)$, the propagation parameters of interest are the propagation constant $\gamma = \alpha + j\beta$ and the complex intrinsic impedance η_c. The general expressions for α, β, and η_c are given by Eqs. (7.66a), (7.66b), and (7.70), respectively, and approximate expressions are given in Table 7-1 for the special cases of low-loss media and good conducting media. If medium 1 is characterized by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 by $(\varepsilon_2, \mu_2, \sigma_2)$, as shown in Fig. 8-7, the expressions for the electric and magnetic fields in media 1 and 2 can be obtained from Eqs. (8.11a) through (8.14a) of Table 8-1 by replacing jk with γ and η with η_c everywhere. Thus,

Medium 1

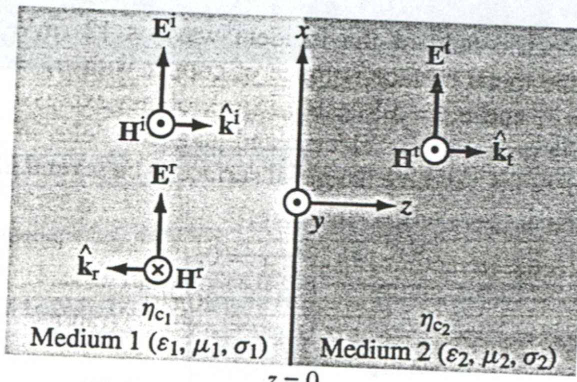
$$\widetilde{\mathbf{E}}_1(z) = \hat{\mathbf{x}} E^{i}_{0}(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}), \qquad (8.22a)$$

$$\widetilde{\mathbf{H}}_1(z) = \hat{\mathbf{y}} \frac{E^{i}_{0}}{\eta_{c_1}}(e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}), \qquad (8.22b)$$

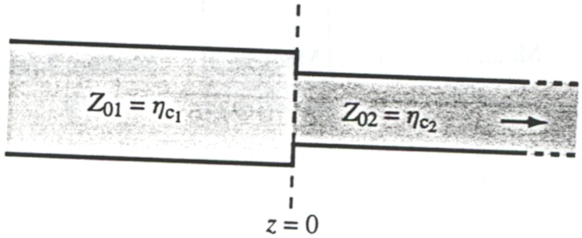
Medium 2

$$\widetilde{\mathbf{E}}_2(z) = \hat{\mathbf{x}} \tau E^{i}_{0} e^{-\gamma_2 z}, \qquad (8.23a)$$

$$\widetilde{\mathbf{H}}_2(z) = \hat{\mathbf{y}} \tau \frac{E^{i}_{0}}{\eta_{c_2}} e^{-\gamma_2 z}, \qquad (8.23b)$$



(a) Boundary between dielectric media



(b) Transmission-line analogue

Figure 8-7: Normal incidence at a planar boundary between two lossy media.

where $\gamma_1 = \alpha_1 + j\beta_1$, $\gamma_2 = \alpha_2 + j\beta_2$, and

$$\boxed{\begin{aligned} \Gamma &= \frac{\eta_{c_2} - \eta_{c_1}}{\eta_{c_2} + \eta_{c_1}}, && (8.24a) \\ \tau &= 1 + \Gamma = \frac{2\eta_{c_2}}{\eta_{c_2} + \eta_{c_1}}. && (8.24b) \end{aligned}}$$

Because η_{c_1} and η_{c_2} are, in general, complex, Γ and τ may be complex as well.

Example 8-3 Normal Incidence on a Metal Surface

A 1-GHz x-polarized TEM wave traveling in the $+z$-direction is incident in air upon a metal surface coincident with the x–y plane at $z = 0$. If the amplitude of the electric field of the incident wave is 12 (mV/m) and the metal surface is made of copper with $\mu_r = 1$, $\varepsilon_r = 1$, and $\sigma = 5.8 \times 10^7$ (S/m), obtain expressions for the instantaneous electric and magnetic fields in the air medium. Assume the metal surface to be several skin depths deep.

Solution: In medium 1 (air), $\alpha = 0$,

$$\beta = k_1 = \frac{\omega}{c} = \frac{2\pi \times 10^9}{3 \times 10^8} = \frac{20\pi}{3} \quad \text{(rad/m)},$$

$$\eta_1 = \eta_0 = 377\ (\Omega), \qquad \lambda = \frac{2\pi}{k_1} = 0.3 \text{ m}.$$

At $f = 1$ GHz, copper is an excellent conductor because

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{5.8 \times 10^7}{2\pi \times 10^9 \times (10^{-9}/36\pi)} = 1 \times 10^9 \gg 1.$$

Use of Eq. (7.77c) gives

$$\eta_{c_2} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= (1+j)\left[\frac{\pi \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}\right]^{1/2}$$

$$= 8.25(1+j) \quad (\text{m}\Omega).$$

Since η_{c_2} is so small compared to $\eta_0 = 377\ (\Omega)$ for air, the copper surface acts, in effect, like a short circuit. Hence,

$$\Gamma = \frac{\eta_{c_2} - \eta_0}{\eta_{c_2} + \eta_0} \simeq -1.$$

Upon setting $\Gamma = -1$ in Eqs. (8.11a) and (8.12a) of Table 8-1, we have

$$\widetilde{\mathbf{E}}_1(z) = \hat{\mathbf{x}} E_0^{\mathrm{i}}(e^{-jk_1z} - e^{jk_1z}) = -\hat{\mathbf{x}} j2E_0^{\mathrm{i}} \sin k_1 z, \qquad (8.25\text{a})$$

$$\widetilde{\mathbf{H}}_1(z) = \hat{\mathbf{y}} \frac{E_0^{\mathrm{i}}}{\eta_1}(e^{-jk_1z} + e^{jk_1z}) = \hat{\mathbf{y}} 2\frac{E_0^{\mathrm{i}}}{\eta_1} \cos k_1 z. \qquad (8.25\text{b})$$

With $E_0^{\mathrm{i}} = 12$ (mV/m), the instantaneous fields corresponding to these phasors are

$$\mathbf{E}_1(z,t) = \mathfrak{Re}[\widetilde{\mathbf{E}}_1(z)\, e^{j\omega t}] = \hat{\mathbf{x}}\, 2E_0^{\mathrm{i}} \sin k_1 z \sin \omega t = \hat{\mathbf{x}}\, 24 \sin(20\pi z/3) \sin(2\pi \times 10^9 t) \quad \text{(mV/m)},$$

$$\mathbf{H}_1(z,t) = \mathfrak{Re}[\widetilde{\mathbf{H}}_1(z)\, e^{j\omega t}] = \hat{\mathbf{y}}\, 2\frac{E_0^{\mathrm{i}}}{\eta_1} \cos k_1 z \cos \omega t = \hat{\mathbf{y}}\, 64 \cos(20\pi z/3) \cos(2\pi \times 10^9 t) \quad (\mu\text{A/m}).$$

Plots of the magnitude of $\mathbf{E}_1(z,t)$ and $\mathbf{H}_1(z,t)$ are shown in Fig. 8-8 as a function of negative z at various values of ωt. The standing-wave patterns exhibit a repetition period of $\lambda/2$, and E and H are in phase quadrature (90° phase shift) in both space and time. This behavior is identical with that of the standing-wave patterns for voltage and current on a shorted transmission line. ■

REVIEW QUESTIONS

Q8.1 What boundary conditions were used in the derivations of the expressions for Γ and τ?

Q8.2 In the radar radome design of Example 8-1, all the incident energy in medium 1 ends up getting transmitted into medium 3, and vice versa. Does this imply that no reflections take place within medium 2? Explain.

Q8.3 Explain on the basis of boundary conditions why it is necessary that $\Gamma = -1$ at the boundary between a dielectric and a perfect conductor.

EXERCISE 8.1 To eliminate wave reflections, a dielectric slab of thickness d and relative permittivity ε_{r_2} is to be inserted between two semi-infinite media with relative permittivities $\varepsilon_{r_1} = 1$ and $\varepsilon_{r_3} = 16$. Use the quarter-wave through consideration of the ray path traversed by the incident, reflected, and transmitted wavefronts.

In view of Eq. (8.54), the boundary conditions given by Eqs. (8.51) and (8.53) reduce to

$$E^{\mathrm{i}}_{\perp 0} + E^{\mathrm{r}}_{\perp 0} = E^{\mathrm{t}}_{\perp 0}, \quad (8.57a)$$

$$\frac{\cos\theta_{\mathrm{i}}}{\eta_1}(-E^{\mathrm{i}}_{\perp 0} + E^{\mathrm{r}}_{\perp 0}) = -\frac{\cos\theta_{\mathrm{t}}}{\eta_2} E^{\mathrm{t}}_{\perp 0}. \quad (8.57b)$$

These two equations can be solved simultaneously to yield the following expressions for the reflection and transmission coefficients in the perpendicular polarization case:

$$\boxed{\Gamma_\perp = \frac{E^{\mathrm{r}}_{\perp 0}}{E^{\mathrm{i}}_{\perp 0}} = \frac{\eta_2\cos\theta_{\mathrm{i}} - \eta_1\cos\theta_{\mathrm{t}}}{\eta_2\cos\theta_{\mathrm{i}} + \eta_1\cos\theta_{\mathrm{t}}}, \quad (8.58a)}$$

$$\boxed{\tau_\perp = \frac{E^{\mathrm{t}}_{\perp 0}}{E^{\mathrm{i}}_{\perp 0}} = \frac{2\eta_2\cos\theta_{\mathrm{i}}}{\eta_2\cos\theta_{\mathrm{i}} + \eta_1\cos\theta_{\mathrm{t}}}. \quad (8.58b)}$$

These two coefficients, which formally are known as the ***Fresnel reflection and transmission coefficients for perpendicular polarization***, are related by

$$\boxed{\tau_\perp = 1 + \Gamma_\perp. \quad (8.59)}$$

If medium 2 is a perfect conductor ($\eta_2 = 0$), Eqs. (8.58a) and (8.58b) reduce to $\Gamma_\perp = -1$ and $\tau_\perp = 0$, respectively, which means that the incident wave is totally reflected by the conducting medium.

For nonmagnetic dielectrics with $\mu_1 = \mu_2 = \mu_0$ and with the help of Eq. (8.56), the expression for $\Gamma_\perp$ can be written as

$$\boxed{\Gamma_\perp = \frac{\cos\theta_{\mathrm{i}} - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_{\mathrm{i}}}}{\cos\theta_{\mathrm{i}} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_{\mathrm{i}}}} \quad (\text{for } \mu_1 = \mu_2). \quad (8.60)}$$

Since $(\varepsilon_2/\varepsilon_1) = (n_2/n_1)^2$, this expression can also be written in terms of the indices of refraction n_1 and n_2.

Example 8-6 Wave Incident Obliquely on a Soil Surface

Using the coordinate system of Fig. 8-15, a plane wave radiated by a distant antenna is incident in air upon a plane soil surface at $z = 0$. The electric field of the incident wave is given by

$$\mathbf{E}^{\mathrm{i}} = \hat{\mathbf{y}}100\cos(\omega t - \pi x - 1.73\pi z) \quad (\mathrm{V/m}), \quad (8.61)$$

and the soil medium may be assumed to be a lossless dielectric with a relative permittivity of 4.

(a) Determine k_1, k_2, and the incidence angle θ_{i}.
(b) Obtain expressions for the total electric fields in air and in the soil medium.
(c) Determine the average power density carried by the wave traveling in the soil medium.

Solution: (a) We begin by converting Eq. (8.61) into phasor form, akin to the expression given by Eq. (8.46a):

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = \hat{\mathbf{y}}100e^{-j\pi x - j1.73\pi z}$$
$$= \hat{\mathbf{y}}100e^{-jk_1 x_{\mathrm{i}}} \quad (\mathrm{V/m}), \quad (8.62)$$

where x_{i} is the axis along which the wave is traveling and

$$k_1 x_{\mathrm{i}} = \pi x + 1.73\pi z. \quad (8.63)$$

Using Eq. (8.47a), we have

$$k_1 x_{\mathrm{i}} = k_1 x\sin\theta_{\mathrm{i}} + k_1 z\cos\theta_{\mathrm{i}}. \quad (8.64)$$

Hence,

$$k_1\sin\theta_{\mathrm{i}} = \pi,$$
$$k_1\cos\theta_{\mathrm{i}} = 1.73\pi,$$

which together give

$$k_1 = \sqrt{\pi^2 + (1.73\pi)^2} = 2\pi \quad (\mathrm{rad/m}),$$
$$\theta_{\mathrm{i}} = \tan^{-1}\left(\frac{\pi}{1.73\pi}\right) = 30^\circ.$$

The wavelength in medium 1 (air) is

$$\lambda_1 = \frac{2\pi}{k_1} = 1 \text{ m},$$

and the wavelength in medium 2 (soil) is

$$\lambda_2 = \frac{\lambda_1}{\sqrt{\varepsilon_{r_2}}} = \frac{1}{\sqrt{4}} = 0.5 \text{ m}.$$

The corresponding wave number in medium 2 is

$$k_2 = \frac{2\pi}{\lambda_2} = 4\pi \quad \text{(rad/m)}.$$

Since $\widetilde{\mathbf{E}}^{\rm i}$ is along $\hat{\mathbf{y}}$, it is perpendicularly polarized ($\hat{\mathbf{y}}$ is perpendicular to the plane of incidence containing the surface normal $\hat{\mathbf{z}}$ and the propagation direction $\hat{\mathbf{x}}_{\rm i}$).

b) Corresponding to $\theta_{\rm i} = 30°$, the transmission angle $\theta_{\rm t}$ s obtained with the help of Eq. (8.56):

$$\sin\theta_{\rm t} = \frac{k_1}{k_2}\sin\theta_{\rm i} = \frac{2\pi}{4\pi}\sin 30° = 0.25$$

r

$$\theta_{\rm t} = 14.5°.$$

ith $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 = \varepsilon_{r_2}\varepsilon_0 = 4\varepsilon_0$, the reflection and nission coefficients for perpendicular polarization termined with the help of Eqs. (8.59) and (8.60),

$$\Gamma_\perp = \frac{\cos\theta_{\rm i} - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_{\rm i}}}{\cos\theta_{\rm i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_{\rm i}}} = -0.38,$$

$$\tau_\perp = 1 + \Gamma_\perp = 0.62.$$

ing Eqs. (8.48a) and (8.49a) with $E^{\rm i}_{\perp 0} = 100$ V/m and $= \theta_{\rm r}$, the total electric field phasor in medium 1 is

$$\begin{aligned}\widetilde{\mathbf{E}}^1_\perp &= \widetilde{\mathbf{E}}^{\rm i}_\perp + \widetilde{\mathbf{E}}^{\rm r}_\perp \\ &= \hat{\mathbf{y}}E^{\rm i}_{\perp 0}e^{-jk_1(x\sin\theta_{\rm i} + z\cos\theta_{\rm i})} \\ &\quad + \hat{\mathbf{y}}\Gamma E^{\rm i}_{\perp 0}e^{-jk_1(x\sin\theta_{\rm i} - z\cos\theta_{\rm i})} \\ &= \hat{\mathbf{y}}100e^{-j(\pi x + 1.73\pi z)} - \hat{\mathbf{y}}38e^{-j(\pi x - 1.73\pi z)},\end{aligned}$$

and the corresponding instantaneous electric field in medium 1 is

$$\begin{aligned}\mathbf{E}^1_\perp(x,z,t) &= \Re\mathfrak{e}\left[\widetilde{\mathbf{E}}^1_\perp e^{j\omega t}\right] \\ &= \hat{\mathbf{y}}[100\cos(\omega t - \pi x - 1.73\pi z) \\ &\quad - 38\cos(\omega t - \pi x + 1.73\pi z)] \quad \text{(V/m)}.\end{aligned}$$

In medium 2, using Eq. (8.49c) with $E^{\rm t}_{\perp 0} = \tau_\perp E^{\rm i}_{\perp 0}$ gives

$$\begin{aligned}\widetilde{\mathbf{E}}^{\rm t}_\perp &= \hat{\mathbf{y}}\tau E^{\rm i}_{\perp 0}e^{-jk_2(x\sin\theta_{\rm t} + z\cos\theta_{\rm t})} \\ &= \hat{\mathbf{y}}62e^{-j(\pi x + 3.87\pi z)}\end{aligned}$$

and, correspondingly,

$$\begin{aligned}\mathbf{E}^{\rm t}_\perp(x,z,t) &= \Re\mathfrak{e}\left[\widetilde{\mathbf{E}}^{\rm t}_\perp e^{j\omega t}\right] \\ &= \hat{\mathbf{y}}62\cos(\omega t - \pi x - 3.87\pi z) \quad \text{(V/m)}.\end{aligned}$$

(c) In medium 2, $\eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} \simeq 120\pi/\sqrt{4} = 60\pi$ (Ω), and the average power density carried by the wave is

$$S^{\rm t}_{\rm av} = \frac{|E^{\rm t}_{\perp 0}|^2}{2\eta_2} = \frac{(62)^2}{2 \times 60\pi} = 10.2 \quad \text{(W/m}^2\text{)}. \blacksquare$$

8-4.2 Parallel Polarization

If we interchange **E** and **H** of the perpendicular polarization situation, while keeping in mind the requirement that relates the directions of **E** and **H** to the direction of propagation for each of the incident, reflected, and transmitted waves, we end up with the geometry shown in Fig. 8-16 for parallel polarization. Now the electric fields lie in the plane of incidence, and the associated magnetic fields are perpendicular to the plane of incidence. With reference to the directions indicated in Fig. 8-16, the fields of the incident, reflected, and transmitted waves are given by

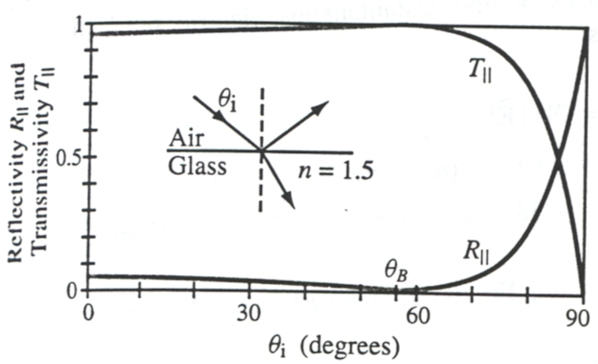


Figure 8-19: Angular plots for $(R_{\|}, T_{\|})$ for an air–glass interface.

Example 8-7 Beam of Light

A 5-W beam of light with circular cross section is incident in air upon a plane boundary of a dielectric medium with an index of refraction of 5. If the angle of incidence is 60° and the incident wave is parallel polarized, determine the transmission angle and the powers contained in the reflected and transmitted beams.

Solution: From Eq. (8.56),

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{1}{5}\sin 60° = 0.17$$

or

$$\theta_t = 10°.$$

With $\varepsilon_2/\varepsilon_1 = n_2^2/n_1^2 = (5)^2 = 25$, the reflection coefficients for parallel polarization can be computed by applying Eq. (8.68) as follows:

$$\Gamma_{\|} = \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}{(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}$$

$$= \frac{-25\cos 60° + \sqrt{25 - \sin^2 60°}}{25\cos 60° + \sqrt{25 - \sin^2 60°}} = -0.435.$$

The reflected and transmitted powers are then

$$P_{\|}^{r} = P_{\|}^{i}|\Gamma_{\|}|^2 = 5(0.435)^2 = 0.95 \text{ W},$$

$$P_{\|}^{t} = P_{\|}^{i} - P_{\|}^{r} = 5 - 0.95 = 4.05 \text{ W}. \blacksquare$$

8-6 Geometric Optics

In the foregoing material, we examined the reflection and refraction behavior of *uniform, plane* electromagnetic waves when incident upon *planar* boundaries between dissimilar materials. In practice, wave interaction with matter may involve waves that are neither uniform nor plane. Also, the objects that the waves interact with may be of any shape and size. How then do we put the results we obtained in the preceding sections to use when dealing with practical situations? The answer depends in large part on the size of the object and the degree of curvature of its surfaces relative to the wavelength of the incident wave. We have available to us two fundamental approaches for describing wave interaction with matter; these are the physical-optics method and the geometric-optics method. In ***physical optics***, also known as ***wave optics***, a wave is characterized mathematically by all its attributes: its amplitude, phase factor, and polarization vector. In contrast, in ***geometric optics***, otherwise known as ***ray optics***, a wave is represented by a ray denoting the direction of travel of the wave's energy. Neither the phase nor the polarization of the wave is accounted for explicitly in geometric optics. Thus, geometric optics is a ray approximation of physical optics. Although it is mathematically superior and its formulation is traceable back to Maxwell's equations, the physical-optics method is mathematically more complicated to apply than the geometric-optics method, and therefore physical optics is used whenever geometric optics is inapplicable or when a greater degree of accuracy is desired than that used in geometric optics. *In general, the geometric-optics method provides fairly accurate results whenever the size of the object illuminated by an electromagnetic wave is much larger than the wavelength λ.* There is also the

Substituting Eq. (8.33) into Eq. (8.34) gives

$$\sin\theta_3 = \left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right)\sin\theta_1 = \sin\theta_1.$$

Hence, $\theta_3 = \theta_1$. The slab displaces the beam's position, but the beam's direction remains unchanged. ■

EXERCISE 8.4 In the visible part of the electromagnetic spectrum, the index of refraction of water is 1.33. What is the critical angle for light waves generated by an upward-looking underwater light source?

Ans. $\theta_c = 48.8°$.

EXERCISE 8.5 If the light source of Exercise 8.4 is situated at a depth of 1 m below the water surface and if its beam is isotropic (radiates in all directions), how large a circle would it illuminate when observed from above?

Ans. Circle's diameter = 2.28 m.

8-3 Fiber Optics

By successive total internal reflections, as indicated in Fig. 8-12(a), light can be guided through thin dielectric rods made of glass or transparent plastic, known as *optical fibers*. Because the light is confined to traveling within the rod, the only loss in power is due to reflections at the sending and receiving ends of the fiber and absorption by the fiber material (because it is not a perfect dielectric). Fiber optics is useful for the transmission of wide-bandwidth signals and in a wide range of imaging applications.

An optical fiber usually consists of a cylindrical *fiber core* with an index of refraction n_f, surrounded by another cylinder of lower index of refraction, n_c, called a *cladding*, as shown in Fig. 8-12(b). The cladding layer serves to optically isolate the fiber from adjacent fibers when a large number of fibers are packed in close proximity, thereby avoiding the leakage of light from one fiber to another. To satisfy the condition of total intern[al] reflection, the incident angle θ_3 in the fiber core must b[e] equal to or greater than the critical angle θ_c for a wav[e] in the fiber medium (with n_f) incident upon the claddi[ng] medium (with n_c). From Eq. (8.32a), we have

$$\sin\theta_c = \frac{n_c}{n_f}. \qquad (8.3$$

To meet the total-reflection requirement that $\theta_3 \geq \theta_c$, [it] is then necessary that $\sin\theta_3 \geq n_c/n_f$. The angle θ_2 is [the] complement of angle θ_3, and $\cos\theta_2 = \sin\theta_3$. Hence, [the] necessary condition may be written as

$$\cos\theta_2 \geq \frac{n_c}{n_f}. \qquad (8.$$

Moreover, θ_2 is related to the incidence angle on the f[ace] of the fiber, θ_i, by Snell's law:

$$\sin\theta_2 = \frac{n_0}{n_f}\sin\theta_i, \qquad (8$$

where n_0 is the index of refraction of the med[ium] surrounding the fiber ($n_0 = 1$ for air and $n_0 =$ 1.33 if the fiber is in water), or

$$\cos\theta_2 = \left[1 - \left(\frac{n_0}{n_f}\right)^2 \sin^2\theta_i\right]^{1/2}. \qquad ($$

Using Eq. (8.38) in the left-hand side of Eq. (8.36 then solving for $\sin\theta_i$ gives

$$\sin\theta_i \leq \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2}. \qquad ($$

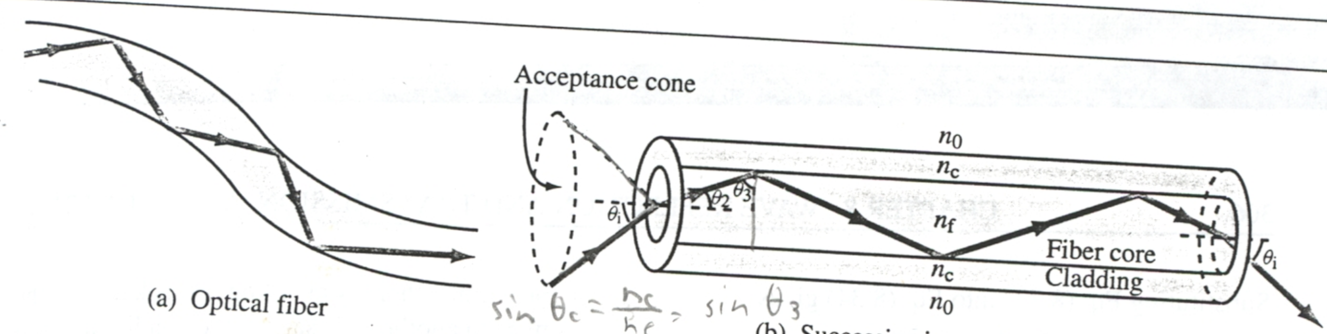
The *acceptance angle* θ_a is defined as the maximum of θ_i for which the condition of total internal refle[ction] remains satisfied:

$$\boxed{\sin\theta_{a_i} \leq \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2}. \qquad (8.40)}$$

$(n_f > n_c)$

$n_c = n_0$ (clad) n_0

$\sin\theta_a' > \sin\theta_a$

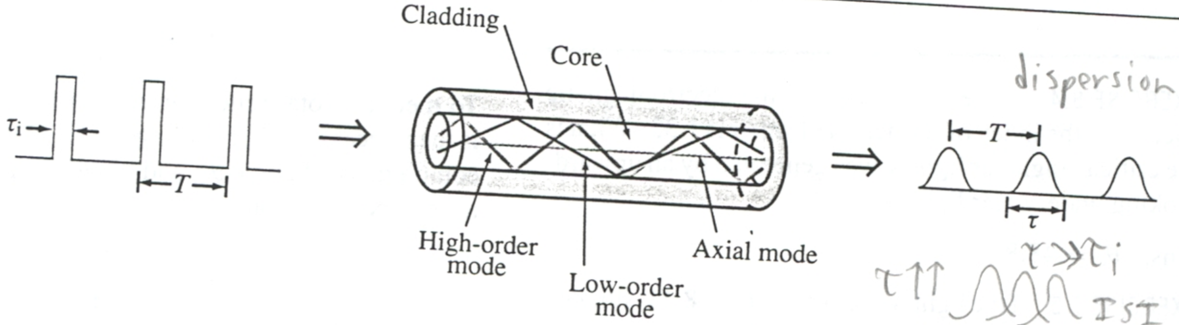


$\sin\theta_c = \frac{n_c}{n_f} = \sin\theta_3$

$n_c \approx n_f \approx 1.5$

Figure 8-12: Waves can be guided along optical fibers as long as the reflection angles exceed the critical angle for total internal reflection.

$\theta_2 = 90 - \theta_3$ $\quad \sin\theta_2 = \frac{n_0}{n_f}\sin\theta_i$



dispersion

$\tau \gg \tau_i$

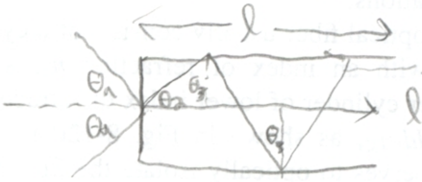
$\tau\uparrow\uparrow$ ISI

Want $\tau < T$ worst case

Figure 8-13: Distortion of rectangular pulses caused by modal dispersion in optical fibers.

$\boxed{\tau < \frac{T}{2}}$

The angle θ_a is equal to half the angle of the acceptance cone of the fiber. Any ray of light incident upon the face of the core fiber at an incidence angle within the acceptance cone can propagate down the core. This means that there can be a larger number of ray paths, called *modes*, by which light energy can travel in the core. Higher-angle rays travel longer paths than rays that propagate along the axis of the fiber, as illustrated by the three modes shown in Fig. 8-13. Consequently, different modes have different transit times between the two ends of the fiber. This property of optical fibers is called *modal dispersion* and has the undesirable effect of changing the shape of pulses used for the transmission of digital data. When a rectangular pulse of light incident upon the face of the fiber gets broken up into many modes and the different modes do not arrive at the other end of the fiber at the same time, the pulse shape gets distorted, both in shape and length. In the example shown in Fig. 8-13, the narrow rectangular pulses at the input side of the optical fiber are of width τ_i separated by a time duration T. After propagating through the fiber core, modal dispersion causes the pulses to look more like spread-out sine waves with spread-out width τ. If the output pulses spread out so much that $\tau > T$, the output signals will smear out, making it impossible to read the transmitted message. Hence, to ensure that the



$l_{long} = \frac{l}{\cos\theta_2}$

transmitted pulses remain distinguishable at the output side of the fiber, it is necessary that τ be shorter than T. As a safety margin, it is common practice to require that $T \geq 2\tau$.

The spread-out width τ is equal to the time delay Δt between the arrival of the slowest ray and the fastest ray. The slowest ray is the one traveling the longest distance and corresponds to the ray incident upon the input face of the fiber at the acceptance angle θ_a. From the geometry of Fig. 8-12(b) and Eq. (8.36), this ray corresponds to $\cos\theta_2 = n_c/n_f$. For an optical fiber of length l, the length of the path traveled by such a ray is

$$l_{max} = \frac{l}{\cos\theta_2} = l\frac{n_f}{n_c}, \qquad (8.41)$$

and its travel time in the fiber at the velocity $u_p = c/n_f$ is

$$t_{max} = \frac{l_{max}}{u_p} = \frac{l n_f^2}{c n_c}. \qquad (8.42)$$

The minimum time of travel is realized by the axial ray and is given by

$$t_{min} = \frac{l}{u_p} = \frac{l}{c_0} n_f. \qquad (8.43)$$

The total time delay is therefore

$n_f \gtrsim n_c \to \tau\downarrow$

$n_f > n_c \to \tau\uparrow$

$$\tau = \Delta t = t_{max} - t_{min} = \frac{l n_f}{c}\left(\frac{n_f}{n_c} - 1\right) \quad \text{(s)}. \qquad (8.44)$$

As we stated before, to retrieve the desired information from the transmitted signals, it is advisable that T, the interpulse period of the input train of pulses, be no shorter than 2τ. This, in turn, means that the data rate (in bits per second), or equivalently the number of pulses per second, that can be transmitted through the fiber is limited to

$$f_p = \frac{1}{T} = \frac{1}{2\tau} = \frac{c n_c}{2 l n_f (n_f - n_c)} \quad \text{(bits/s)}. \qquad (8.45)$$

$n_f \gtrsim n_c \to \begin{cases} f_p \text{ large} \\ \theta_a \text{ small} \end{cases}$

Example 8-5 Transmission Data Rate on Optical Fibers

A 1-km-long optical fiber (in air) is made of a fiber core with an index of refraction of 1.52 and a cladding with an index of refraction of 1.49. Determine
(a) the acceptance angle θ_a, and
(b) the maximum usable data rate that can be transmitted through the fiber.

Solution: (a) From Eq. (8.40),

$$\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2} = [(1.52)^2 - (1.49)^2]^{1/2} = 0.3,$$

which corresponds to $\theta_a = 17.5°$.

(b) From Eq. (8.45),

$$f_p = \frac{c n_c}{2 l n_f (n_f - n_c)} = \frac{3 \times 10^8 \times 1.49}{2 \times 10^3 \times 1.52(1.52 - 1.49)} = 4.9 \text{ (Mb/s)}.$$

EXERCISE 8.6 If the index of refraction of the cladding material in Example 8-5 is increased to 1.50, what would be the new maximum usable data rate?

Ans. 7.4 (Mb/s).

8-4 Wave Reflection and Transmission at Oblique Incidence

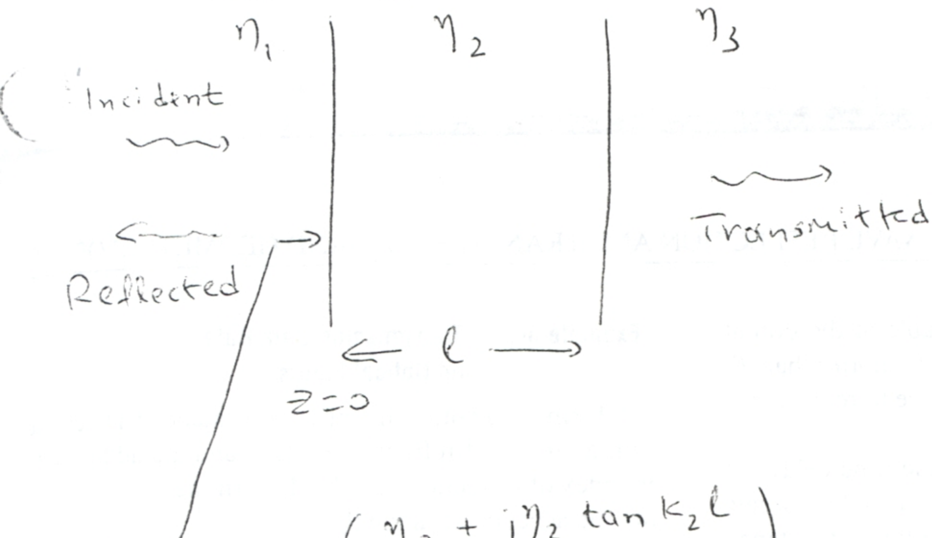
For normal incidence, the reflection coefficient Γ and transmission coefficient τ of a boundary between two different media is independent of the polarization of the incident wave, because the electric and magnetic fields of a normally incident plane wave are both always tangential to the boundary regardless of the wave polarization. This is not the case for oblique incidence at an angle

Reflection Problems with Several Dielectrics

(1)

ECE 3065

LECTURE 12



$$Z_{L_1} = \eta_2 \left(\frac{\eta_3 + j\eta_2 \tan k_2 \ell}{\eta_2 + j\eta_3 \tan k_2 \ell} \right)$$

$$\rho = \frac{Z_{L_1} - \eta_1}{Z_{L_1} + \eta_1}$$

(A) Half-wave Diel. Window

$$\eta_1 = \eta_3 , \quad k_2 \ell = m\pi \Rightarrow Z_L = \eta_3 = \eta_1 \rightarrow \rho = 0$$

(-) Reflections for frequencies other than that for which its thickness is a multiple of $\lambda/2$

(B) Electrically Thin window

$$\eta_1 = \eta_3 , \quad k_2 \ell \ll 1 \Rightarrow \tan(k_2 \ell) \simeq k_2 \ell$$

$$Z_{L_1} \cong \eta_2 \left(\frac{\eta_1 + j\eta_2 k_2 \ell}{\eta_2 + j\eta_1 k_2 \ell} \right) \simeq \eta_1 \left[1 + jk_2 \ell \left(\frac{\eta_2}{\eta_1} - \frac{\eta_1}{\eta_2} \right) \right]$$

$$\rightsquigarrow \rho \simeq j \frac{k_2 \ell}{2} \left(\frac{\eta_2}{\eta_1} - \frac{\eta_1}{\eta_2} \right) \sim k_2 \ell \sim \text{length}$$

(Power reflected $\approx$ length^2)

(e.g.) Polystyrene ($\varepsilon_r \simeq 2.54$), 3mm thick at normal incidence at 3 GHz from air $\Rightarrow \rho = -j0.145$ (2% reflected power

Quarter-wave Coating for Eliminating Reflections

$k_2 l = \frac{\pi}{2}$, $\eta_2 = \sqrt{\eta_1 \eta_3}$ $\Rightarrow Z_{L_1} = \frac{\eta_2^2}{\eta_3} = \eta_1$ $(\rho = 0)$

The matching is perfect only at specific frequencies for which the length is an odd multiple of $\lambda/4$, but is approximately correct for bands of frequencies about these values

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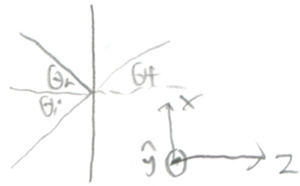


Oblique ~~Impedance~~ Incidence

$(Z_z)_{TM} = \eta \cos\theta$, $(Z_z)_{TE} = \eta \sec\theta = \eta / \cos\theta$

$\left(= \frac{E_{x+}}{H_{y+}} = - \frac{E_{x-}}{H_{y-}} \right)$

$\eta_1 \cos\theta_i$ | $\eta_2 \cos\theta_t$ TM

$\eta_1 / \cos\theta_i$ | $\eta_2 / \cos\theta_t$ TE

{ Reflection and Transmission of a Circularly Polarized wave at oblique Incidence }

A circularly polarized wave can be resolved into two linearly polarized parts $\left(\bar{E} = (\hat{x} \pm j\hat{y}) E_1 e^{-jkz} \right)$ TE and TM combo.

Assume wave incidence at $\theta_i = 60^\circ$ from air onto fused quartz with $\varepsilon_r = 3.78$

Snell's: $\theta_2 = \sin^{-1}\left[\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin\theta_1\right] = \sin^{-1} \frac{0.866}{\sqrt{3.78}} = 26.4^\circ$

TM component

$Z_{z_1} = \eta_1 \cos\theta_1 = 377 \cos 60^\circ = 188.5\ \Omega$

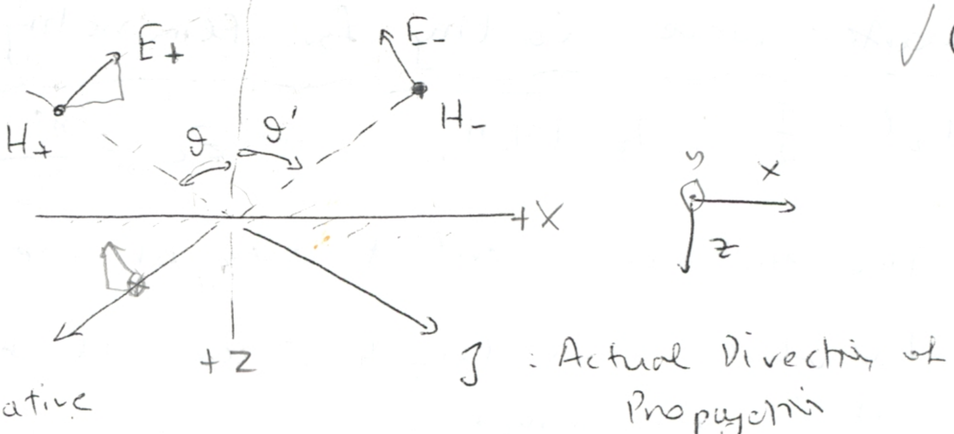
$Z_{z_2} = \eta_2 \cos\theta_2 = \frac{377}{\sqrt{3.78}} \cos 26.4^\circ = 174\ \Omega$

$\rho = \frac{174 - 188.5}{174 + 188.5} = -0.04$, $\tau = 0.96$

Normal incidence → transmit circularly polarized wave. elliptical

TE component $Z_{z_1} = \eta_1 \sec\theta_1 = 756\ \Omega$

(TM)



$$\zeta = x \sin\vartheta + z\cos\vartheta, \quad \zeta' = -x\sin\vartheta' + z\cos\vartheta'$$

$$E_x(x,z) = E_+ \cos\vartheta\, e^{-jk(x\sin\vartheta + z\cos\vartheta)} - E_- \cos\vartheta'\, e^{jk(-x\sin\vartheta' + z\cos\vartheta')}$$

$$E_z(x,z) = -E_+ \sin\vartheta\, e^{-jk(x\sin\vartheta + z\cos\vartheta)} - E_- \sin\vartheta'\, e^{jk(-x\sin\vartheta' + z\cos\vartheta')}$$

$$H_y(x,z) = H_+ e^{-jk(x\sin\vartheta + z\cos\vartheta)} + H_- e^{jk(-x\sin\vartheta' + z\cos\vartheta')}$$

?

Two phase constants.

$$\beta_x = k\sin\vartheta, \quad \beta_z = k\cos\vartheta$$

$$\bar{E}_+(x,z,t) = \mathrm{Re}\left[\tilde{E}_+ e^{j(\omega t - \beta_x x - \beta_z z)}\right]$$

Phase velocity : $v_{px} = \dfrac{\omega}{\beta_x} = \dfrac{v}{\sin\vartheta}$ ($\omega t - \beta_x x$: constant)

$v_{pz} = \dfrac{\omega}{\beta_z} = \dfrac{v}{\cos\vartheta}$ ($\omega t - \beta_z z$: constant)

No problem with $v_{px}, v_{pz} > c_0 = v$

↑

velocity of fictitious point of intersection of the wavefront and a line drawn in the selected direction