ECE 3065

Plane-Wave Propagation in Lossy Media

Conducting medium $\Rightarrow \sigma \neq 0$

$$\nabla^2 \tilde{E} - \gamma^2 \tilde{E} = 0$$

with: $\gamma^2 = -\omega^2 \mu \varepsilon_c = -\omega^2 \mu (\varepsilon' - j\varepsilon'')$

$$\varepsilon' = \varepsilon, \quad \varepsilon'' = \sigma/\omega$$

$$\gamma = \alpha + j\beta$$

attenuation constant — phase constant

$$(\alpha + j\beta)^2 = (\alpha^2 - \beta^2) + j2\alpha\beta = -\omega^2 \mu \varepsilon' + j\omega^2 \mu \varepsilon''$$

$$\Rightarrow \begin{cases} \alpha = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right\}^{1/2}, & (\text{Np/m}) \\ \beta = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right\}^{1/2} & (\text{rad/m}) \end{cases}$$

valid for any medium

Assume plane wave: $\tilde{E} = \hat{x} \tilde{E}_x(z)$ $\xrightarrow{+\hat{z}}$

$$\frac{d^2 \tilde{E}_x(z)}{dz^2} - \gamma^2 \tilde{E}_x(z) = 0 \Rightarrow \tilde{E}(z) = \hat{x} \tilde{E}_x(z) = \hat{x} E_{x0} e^{-\gamma z} = \hat{x} E_{x0} e^{-\alpha z} e^{-j\beta z}$$

$\tilde{H} = \frac{1}{\eta_c} (\hat{k} \times \tilde{E})$, η_c = intrinsic impedance of lossy medium

$$\tilde{H}(z) = \hat{y} \tilde{H}_y(z) = \hat{y} \frac{\tilde{E}_x(z)}{\eta_c} = \hat{y} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \quad (\Omega)$$

η_c: complex number $\Rightarrow \tilde{E}(z), \tilde{H}(z)$ NOT IN PHASE for lossy materials

$$|\tilde{E}_x(z)| = |E_{xo} e^{-\alpha z} e^{-j\beta z}| = |E_{xo}| e^{-\alpha z}$$

$\downarrow$

decreases exponentially with z at a rate specified by attenuation constant α.

$|\tilde{H}_y(z)| \sim e^{-\alpha z}$

Due to the conduction of the medium, energy $\rightsquigarrow$ heat.

Field attenuation by $e^{-1} \approx 0.37$ of the value at $z=0$

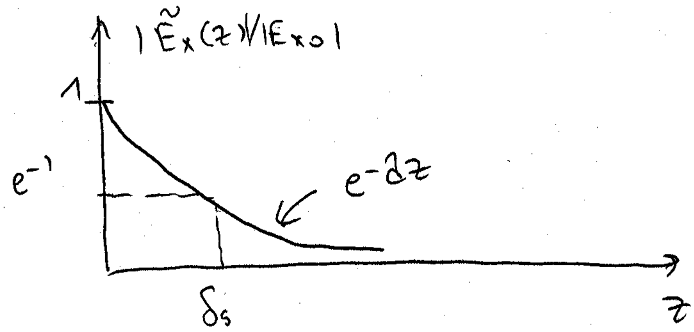
at a distance $z = \delta_s = \frac{1}{\alpha}$ (m)

$\uparrow$

skin depth of the medium

$\uparrow$

shows how well the EM wave penetrates the conducting medium



(·) Perfect dielectric $\Rightarrow \sigma = 0 \Rightarrow \alpha = 0 \Rightarrow \delta_s = \infty$ (no loss for indefinite distance)

(·) Perfect conductor $\Rightarrow \sigma = \infty \xrightarrow[\varepsilon'' = \frac{\sigma}{\omega}]{} \alpha = \infty \Rightarrow \delta_s = 0$

To prevent the energy inside the transmission cables from leaving outward, as well as to shield against the penetration of outside EM energy into the cable, the outer conductor has to be several skin depths thick.

Lossless case: $\sigma = 0 \Rightarrow \alpha = 0,\ \beta = k = \omega\sqrt{\mu\varepsilon}\ ,\ \eta_c = \eta = \sqrt{\frac{\mu}{\varepsilon}}$

Lossy medium: $\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$ to very important

In a perfect dielectric (e.g. free space) → $\sigma = 0 \to \delta_s = \infty$

In a perfect conductor ($\sigma = \infty$) → $\delta_s = 0$

(Insulations have to be many skin depths thick!!)

when $\frac{\varepsilon''}{\varepsilon'} \ll 1$ ($< 10^{-2}$) → low-loss dielectric

$$\alpha \simeq \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad (\text{Np/m})$$

$$\beta \simeq \omega\sqrt{\mu\varepsilon'} = \omega\sqrt{\mu\varepsilon} \quad (\text{rad/m})$$

$$\eta_c \cong \sqrt{\frac{\mu}{\varepsilon}} \quad (\text{Real})$$

$$u_p = \omega/\beta \quad (\text{m/sec})$$

$$\lambda = 2\pi/\beta = u_p/f \quad (\text{m})$$

$$\gamma = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2} \cong j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{2\varepsilon'}\right)$$

$(1-x)^{1/2} \simeq 1 - x/2$

when $\frac{\varepsilon''}{\varepsilon'} \gg 1$ ($> 10^{2}$) → good conductor

$$\alpha = \sqrt{\pi f \mu \sigma} \quad (\text{Np/m})$$

$$\beta = \sqrt{\pi f \mu \sigma} \quad (\text{rad/m})$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma} \quad (\Omega) \quad (\text{complex number } \angle 45^\circ)$$

$$u_p = \sqrt{4\pi f/\mu\sigma} \quad (\text{m/sec})$$

$$\lambda = u_p/f \quad (\text{m})$$

Perfect conductor
$\sigma = \infty \to \alpha = \beta = \infty, \eta_c = 0$
Equivalent to short circuit

when $\frac{\varepsilon''}{\varepsilon'} \in [10^{-2}, 10^{2}]$ → quasi-conductor

Use formulas of (4)

Power Density

Poynting Vector: $\bar{S} = \bar{E} \times \bar{H}$ (W/m²)

Average Power Density: $\tilde{S}_{av} = \frac{1}{2} Re[\tilde{E} \times \tilde{H}^*]$ (W/m²)

Lossless Medium: $\tilde{S}_{av} = \hat{z} \frac{|\tilde{E}|^2}{2\eta}$ (for $+\hat{z}$ propag. wave)

Lossy Medium: $\tilde{S}_{av} = \hat{z} \frac{|E_0|^2}{2|\eta_c|} e^{-2\alpha z} \cos\theta_\eta$

$(\eta_c = |\eta_c| e^{j\theta_\eta})$

$|E_0|$ = magnitude of $\tilde{E}(z)$ at $z=0$

At $z = \delta_s = 1/\alpha \Rightarrow$ attenuation of power $e^{-2} \approx 14\%$

Decibels

$$G\,[db] = 20 \log\left(\frac{E_1}{E_2}\right) = 10 \log\left(\frac{P_1}{P_2}\right)$$

Example

Seawater: $\varepsilon_r = 80$, $\mu_r = 1$, $\sigma = 4$ S/m

At $z=0$: $\bar{H}(0,t) = \hat{y}\, 100 \cos(2\pi \times 10^3 t + 15°)$ A/m

and propagates $\longrightarrow +\hat{z}$

$(\tilde{E} = -\eta_c \hat{k} \times \tilde{H}) \Rightarrow \tilde{E}(z) = \hat{x} E_{x0} e^{-\alpha z} e^{-j\beta z}$

$\tilde{H}(z) = \hat{y} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}$

$= 2\pi \times 10^3 \Rightarrow \omega = 2\pi f \Rightarrow f = 10^3$ Hz $= 1$ kHz

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4}{2\pi \times 10^3 \times 80 \times 10^{-9}/36\pi} = 9 \times 10^5 > 10^2$$

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$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{\pi\, 10^3\, 4\pi 10^{-7}\, 4} = 0.126 \text{ Np/m}$$

$$\beta = \sqrt{\pi f \mu \sigma} = 0.126 \text{ rad/m}$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma} = 0.044\, e^{j\pi/4} \ (\Omega)$$

$$\bar{E}(z,t) = \mathrm{Re}\left[\hat{x}\,|E_{xo}|\, e^{j\phi_o} e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\right]$$

$$= \hat{x}\,|E_{xo}|\, e^{-0.126 z} \cos(2\pi \times 10^3 t - 0.126 z + \phi_o) \quad (\text{V/m})$$

$$\bar{H}(z,t) = \mathrm{Re}\left[\hat{y}\, \frac{|E_{xo}|\, e^{j\phi_o}}{0.044\, e^{j\pi/4}}\, e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\right]$$

$$= \hat{y}\, 22.5\, |E_{xo}|\, e^{-0.126 z} \cos(2\pi \times 10^3 t - 0.126 z + \phi_o - 45^\circ) \quad \text{A/m}$$

At $z=0 \Rightarrow \bar{H}(0,t) = \hat{y}\, 22.5\, |E_{xo}| \cos(\ )$

$$= \hat{y}\, 100 \cos(\ )$$

$$|E_{xo}| = 4.44 \text{ mV/m}$$

$$\phi_o - 45^\circ = 15^\circ \rightarrow \phi_o = 60^\circ$$

$$E(z,t) = \hat{x}\, 4.44\, e^{-0.126 z} \cos(2\pi 10^3 t - 0.126 z + 60^\circ) \ (\text{mV/m})$$

$$\bar{H}(z,t) = \hat{y}\, 100\, e^{-0.126 z} \cos(2\pi 10^3 t - 0.126 z + 15^\circ) \ (\text{mA/m})$$

$$\eta_c = 0.044 \angle 45^\circ \ (\Omega)$$

$$\tilde{S}_{av}(z) = \hat{z}\, \frac{|E_o|^2}{2|\eta_c|}\, e^{-2\alpha z} \cos\theta_\eta$$

$$= \hat{z}\, \frac{(4.44 \times 10^{-3})^2}{2 \times 0.044}\, e^{-0.252 z} \cos 45^\circ$$

$$= \hat{z}\ 0.16\, e^{-0.252 z} \ (\text{mW/m}^2)$$